The big five theorems of elementary functional analysis

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Theorem 1 (Hahn–Banach theorem, or HBT).

Let $(X, \|\cdot\|)$ be a real normed vector space, V a subspace of X, $p\colon X \to [0, \infty)$ a sublinear function, and $\ell\colon V \to \mathbb{R}$ a linear functional such that $\ell \leq (p \restriction V)$. Then there exists an extension $L\colon X \to \mathbb{R}$ of ℓ such that L is linear, $(L \restriction V) \equiv \ell$, and $L \leq p$.

Proof. Zorn's lemma. □

Theorem 2 (Uniform boundedness principle, or UBP).

Let $(X, \|\cdot\|)$ be a real Banach space, and let $\mathcal{S} = \{\ell_i\}_{i \in I}$ be a collection of linear functionals on X, i.e. $\mathcal{S} \subseteq X^*$. If $\mathcal{S}$ is pointwise bounded, then $\mathcal{S}$ is uniformly bounded in $\|\cdot\|_{X^*}$. More specifically, if for all $x \in X$, there exists $C_x < \infty$ such that $|\ell_i(x)| \leq C_x$ for all $i \in I$, then there exists a constant B for which $\|\ell_i\|_{X^*} \leq B$ for all $i \in I$.

Theorem 3 (Open mapping theorem, or OMT).

Let X and Y be Banach spaces, and let $T\colon X \to Y$ be a bounded surjective linear mapping. Then T is an open map: for any open ball $B \subseteq X$, its image $T(B)$ is open in Y.

Theorem 4 (Closed graph theorem, or CGT).

Let X and Y be normed vector spaces, and let $T\colon X \to Y$ be a bounded linear mapping. Then the graph of T, $\{(x, T(x)) : x \in X\}$, is a closed subset of $X \times Y$ with the product topology.

Proof. An easy consequence of the OMT. We can check that the norm $\|(x, T(x))\|_G = \|x\|_X + \|T(x)\|_Y$ makes $G = \text{graph}(T)$ a Banach space. Then consider the inverse of the bounded bijective linear map $(x, T(x)) \twoheadrightarrow x$; the inverse is bounded by the OMT. □

Theorem 5 (Alaoglu's theorem).

Let $(X, \|\cdot\|)$ be a normed vector space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Then the closed unit ball of the dual space $\{\ell \in X^* : \|\ell\|_{X^*} \leq 1\}$ is compact with respect to the weak-* topology.

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