The construction of a Vitali set

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Not all subsets of $\mathbb{R}$ have a well-defined notion of length, or Lebesgue measure. The existence of such a pathological *nonmeasurable set* is one of the unintuitive consequences of assuming the axiom of choice.

Write $I = [0, 1]$. Consider the set of equivalence classes of $I \bmod (\mathbb{Q}, +)$. In other words, consider $I/\sim$, where $x \sim y$ iff $x - y \in \mathbb{Q}$. Let $E \subseteq I$ consist of one representative chosen from each equivalence class in I, which we can define by the axiom of choice. Observe that E is uncountable, and

$$[0, 1] = \bigcup_{q \in \mathbb{Q} \cap [0,1]} (E + q) \bmod 1$$

is a countable partition of I into translations of E. Note that $E + q = \{x + q : x \in E\}$, and length$(E) =$ length$(E + q)$ for any $q \in \mathbb{Q}$ by the translation-invariance of Lebesgue measure. Let us check that this is indeed a partition: we claim that $E + q_1$ and $E + q_2$ are disjoint whenever $q_1 \neq q_2$. Suppose not; then there exists $x + q_1 = y + q_2$, $x, y \in E$, and in particular $x \sim y$. But $q_1 \neq q_2$ implies that $x \neq y$, and by our construction of E, x and y cannot belong to the same equivalence class — contradiction.

Thus, by the countable additivity of the Lebesgue measure m on $[0, 1]$, we have

$$\mathrm{m}([0, 1]) = \sum_{q \in \mathbb{Q} \cap [0,1]} \mathrm{m}(E + q \bmod 1) = \sum_{q \in \mathbb{Q} \cap [0,1]} \mathrm{m}(E).$$

We claim that the **Vitali set** E is not m-measurable. If $\mathrm{m}(E) = 0$, then $\mathrm{m}([0, 1]) = 0 \neq 1$. However, if $\mathrm{m}(E) > 0$, then $\mathrm{m}([0, 1]) = +\infty$, again contradicting $\mathrm{m}([0, 1]) = 1$. Therefore we cannot assign $\mathrm{m}(E)$ in a way that preserves all of translation-invariance, countable additivity, and $\mathrm{m}([0, 1]) = 1$.

The above is also the same proof that there is no uniform probability measure on a countable set $X \ni x_*$: $\mathbb{P}(X) = \sum_{x \in X} \mathbb{P}(\{x\}) = \sum_{x \in X} \mathbb{P}(\{x_*\}) = 1$ means $\mathbb{P}(\{x_*\})$ cannot be zero or nonzero, i.e. cannot be defined at all. Informally, there can be no set whose measure is "$\frac{1}{\infty}$."

The existence of a nonmeasurable set motivates our restriction of the domain of a measure to a σ-algebra instead of the full power set. A more complex example can be found as part of the *Banach–Tarski paradox*.

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