Determining a measure

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This note pertains in part to the uniqueness result of Carathéodory's extension theorem.

How can we determine a measure μ on Σ? Obviously, if we provide a whole σ-algebra's worth of information, i.e. the value of $\mu(A)$ for every $A \in \Sigma$, then we uniquely specify μ. But, measurable sets in Σ do not admit explicit constructive descriptions in general, so defining $\mu(A)$ for a general $A \in \Sigma$ can be quite difficult. If we can instead specify μ on a smaller class of sets $\mathcal{C} \subseteq \Sigma$ to somehow uniquely determine all of μ using countable additivity, then we will have a much easier time constructing measures.

Theorem 1 (σ-finite measures are uniquely determined by their value on a generating π-system)**.**

Let μ_1, μ_2 be σ-finite measures on a σ-algebra Σ that agree on a π-system generating Σ, i.e. $\mu_1(A) = \mu_2(A)$ for all $A \in \Pi$. Then $\mu_1 \equiv \mu_2$ on all of Σ.

Definition 1 (π-system)**.**

A family is a π-**system** if it is closed under finite intersection.

Definition 2 (σ-finite measure)**.**

A measure μ on (Ω, Σ) is σ-**finite** if there exists a countable partition $\Omega_1, \Omega_2, \ldots$ of Ω such that $\mu(\Omega_i) < \infty$ for all $i \geq 1$.

Theorem 1 has some interesting consequences. In probability, *independence* is determined by generating π-systems as well, furthering the connections between intersections, products, and orthogonality. And, because *semialgebras* are π-systems, σ-finite measures defined on a semialgebra are uniquely determined! Thus, the Lebesgue measure on $\mathbb{R}^n$, as we constructed it, is truly unique.

We remark that it suffices to prove Theorem 1 for the case of finite measures, for which $\mu(\Omega) < \infty$. If the result holds in the finite case, then for σ-finite μ and $A \in \Sigma$,

$$\mu_1(A) = \sum_{i=1}^{\infty} \mu_1(A \cap \Omega_i) = \sum_{i=1}^{\infty} \mu_2(A \cap \Omega_i) = \mu_2(A)$$

using the fact that $(\Omega_i)_{i=1}^{\infty}$ partition Ω, the countable additivity of μ_1, μ_2, and the finiteness of μ_1, μ_2 restricted to each Ω_i. Alternative, we can consider $\bigcup_{i=1}^{n} \Omega_i \uparrow \Omega$, where each $\bigcup_{i=1}^{n} \Omega_i$ has finite measure.

Proposition 1 (Properties of a measure)**.**

Let μ be any measure on Σ. Then the following hold.

a. *Monotonicity.* If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

b. *Set difference.* If $\mu(A) < \infty$ and $A \subseteq B$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

c. *Continuity from below.* If $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence of sets with limit $\bigcup_{i=1}^{\infty} A_i$, then

$$\mu(A_i) \uparrow \mu\left(\bigcup_{i=1}^{\infty} A_i\right).$$

d. *Continuity from above.* If $A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing sequence tending to $\bigcap_{i=1}^{\infty} A_i$, and $\mu(A_i) < \infty$ for some $i \geq 1$ (without loss of generality $i = 1$), then

$$\mu(A_i) \downarrow \mu\left(\bigcap_{i=1}^{\infty} A_i\right).$$

We will not prove the properties above, but we will note the finiteness assumptions (which allows the subtraction of measures without the issue of $\infty - \infty$), and the fact that $\mu(B \setminus A)$, $\mu(\bigcup_{i=1}^{\infty} A_i)$, and $\mu(\bigcap_{i=1}^{\infty} A_i)$ are completely determined by the values of $\mu(A)$, $\mu(B)$, and $\mu(A_i)$ given. In contrast, $\mu(A \cup B)$, which equals $\mu(A) + \mu(B) - \mu(A \cap B)$ by the principle of inclusion-exclusion, is not determined by merely $\mu(A)$ and $\mu(B)$. Thus, if we specify $\mu(A)$ for all $A \in \mathcal{C}$, then we also specify the measures of set differences $B \setminus A$ and monotone limits, but not finite intersections. This motivations the following definition and result.

Definition 3 (λ-system)**.**

A family Λ is a λ-**system** if it satisfies the following.

1. *Nonempty.* $\Omega \in \Lambda$.

2. *Closure under set difference.* If $A, B \in \Lambda$ and $A \subseteq B$, then $B \setminus A \in \Lambda$.

3. *Closure under increasing limits.* If $A_1, A_2, \ldots \in \Lambda$ and $A_1 \subseteq A_2 \subseteq \cdots$, then $\bigcup_{i=1}^{\infty} A_i \in \Lambda$.

Proposition 2 (Two measures agree on a λ-system)**.**

Let μ_1, μ_2 be (σ-)finite on Σ. Then $\{A \in \Sigma : \mu_1(A) = \mu_2(A)\}$ is a λ-system.

In some sense, being a π-system and being a λ-system are "complementary" properties; if two measures also agree for all finite intersections, then, as noted above, they should also agree *everywhere*, i.e. on the full σ-algebra. The following result justifies this observation when applied to $\mathcal{C} = \{A \in \Sigma : \mu_1(A) = \mu_2(A)\}$, proving Theorem 1.

Proposition 3 (π-λ)**.**

A family is a σ-algebra iff it is both a π-system and λ-system.

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