Tychonoff's theorem and Kelley's theorem

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2023-01-11

Theorem 1 (Tychonoff's theorem)**.**

Let $\{(X_i, \tau_i)\}_{i \in I}$ be any collection of compact topological spaces. Then $\prod_{i \in I} X_i$ is compact with respect to the product topology.

Tychonoff's theorem is one of the classical results of general topology, often considered to be *the* most important, and its proof is typically the most difficult proof covered in an introductory topology course. In this note, we will present a proof that Tychonoff's theorem is equivalent to the Axiom of Choice (AC, or AoC).

Definition 1 (Compactness)**.**

A topological space (X, τ) is **compact** if every open cover of X has a finite subcover. (An open cover of X is a collection of open sets $\mathcal{C} \subseteq \tau$ such that $X = \bigcup_{O \in \mathcal{C}} O$; a subcover of $\mathcal{C}$ is any cover $\mathcal{C}' \subseteq \mathcal{C}$.)

Definition 2 (Finite intersection property)**.**

A family $\mathcal{F}$ of subsets of X has the **finite intersection property** iff for every finite subcollection $\mathcal{F}' \subseteq \mathcal{F}$ of nonempty subsets, $\bigcap_{E \in \mathcal{F}'} E \neq \varnothing$.

Proposition 1 (Equivalent formulation of compactness)**.**

(X, τ) is compact iff for any collection of closed subsets $\mathcal{C}$ with the finite intersection property, $\bigcap_{C \in \mathcal{C}} C \neq \varnothing$.

Proof. First, observe that by De Morgan's laws, $\mathcal{O} \subseteq \tau$ is an open cover of X iff

$$\bigcap_{O \in \mathcal{O}} (X \setminus O) = X \setminus \bigcup_{O \in \mathcal{O}} O = \varnothing.$$

The contrapositive states that if $\bigcap_{C \in \mathcal{C}} C = \varnothing$, then there exists a finite subcollection $\mathcal{C}' \subseteq \mathcal{C}$ with $\bigcap_{C \in \mathcal{C}'} C = \varnothing$. If X is compact, then any such $\mathcal{C}$ has $\mathcal{O} = \{C^c : C \in \mathcal{C}\}$ an open cover of X, and a finite subcover $\mathcal{O}' \subseteq \mathcal{O}$ has $\mathcal{C}' = \{O^c : O \in \mathcal{O}'\}$ with $\bigcap_{C \in \mathcal{C}'} C = \varnothing$. Conversely, for an open cover $\mathcal{O}$ of X, then $\mathcal{C} = \{O^c : O \in \mathcal{O}\}$ satisfies $\bigcap_{C \in \mathcal{C}} C = \varnothing$ and has a finite subcollection $\mathcal{C}' \subseteq \mathcal{C}$ such that $\{C^c : C \in \mathcal{C}'\} \subseteq \mathcal{O}$ is a finite subcover of X, which shows that X is compact. $\square$

Definition 3 (Axiom of Choice)**.**

Let $\{X_i\}_{i\in I}$ be any collection of nonempty sets. Then the Cartesian product $\prod_{i\in I} X_i$ is nonempty.

Lemma 1 (Zorn's lemma)**.**

Let $\mathcal{A}$ be any nonempty collection of sets. If every chain $\mathcal{C} \subseteq \mathcal{A}$ has an upper bound $Y \in \mathcal{A}$, such that $X \subseteq Y$ for all $X \in \mathcal{C}$, then $\mathcal{A}$ has a $\subseteq$-maximal element. (A *chain* is a subset totally ordered by $\subseteq$.)

Proposition 2.

Zorn's lemma is equivalent to the Axiom of Choice.

We will leave a proof of Proposition 2, and the many equivalencies and consequences of the Axiom of Choice, to a course on set theory. Instead, we present one last lemma we will need in the proof of Theorem 1:

Lemma 2 (Equivalent condition for being in the closure)**.**

Let (X, τ) be a topological space, $x \in X$, and $A \subseteq X$. Then $x \in \overline{A}$ iff every open (basic) set O containing x intersects A, i.e. $O \cap A \neq \varnothing$. (Note that every open set contains some basis set, and $B \cap A \subseteq O \cap A$.)

Proof. If there is an open $O \ni x$ with $O \cap A \neq \varnothing$, then $A \subseteq O^c$ closed, and $\overline{A} \subseteq O^c$. But $x \notin O^c$ means $x \notin \overline{A}$. Conversely, $x \notin \overline{A}$ means $(\overline{A})^c$ is an open set containing x disjoint from A: $(\overline{A})^c \cap A \subseteq (\overline{A})^c \cap \overline{A} = \varnothing$. □

The proof of Theorem 1 proceeds on the following page.

Proof of Tychonoff's theorem. Let $\mathcal{C}$ be a collection with the finite intersection property of closed subsets of X. By Proposition 1, it suffices to show that $\bigcap_{C \in \mathcal{C}} C$ is nonempty.

1. Let Ω be the collection of families of subsets of X that contain $\mathcal{C}$ and have the finite intersection property. We claim that Ω satisfies the hypothesis of Zorn's lemma.

   Let $\mathscr{C} \subseteq \Omega$ be a chain, and let $\mathcal{U} = \bigcup_{\mathcal{F} \in \mathscr{C}} \mathcal{F}$. We wish to show that $\mathcal{U} \in \Omega$.

   a. Since any $\mathcal{F} \in \Omega$ has $\mathcal{F} \supseteq \mathcal{C}$, we have $\mathcal{U} \supseteq \mathcal{C}$ as well.

   b. To check that $\mathcal{U}$ has the finite intersection property, let $\{E_k\}_{k=1}^n \subseteq \mathcal{U}$ be a finite subcollection. There exists $\mathcal{F}_k \in \mathscr{C} \subseteq \Omega$ containing E_k for each $k = 1, \ldots, n$ by definition of $\mathcal{U}$. As $\mathscr{C}$ is linearly ordered, let $\mathcal{F}$ be the maximum of the $\mathcal{F}_k$, so that $\{E_k\}_{k=1}^n \subseteq \mathcal{F} \in \mathscr{C}$. But $\mathcal{F} \in \Omega$ as well, so $\mathcal{F}$ has the finite intersection property, and $\bigcap_{k=1}^n E_k \neq \varnothing$.

   Thus every chain $\mathscr{C} \subseteq \Omega$ has an upper bound $\mathcal{U}$ which belongs to Ω.

2. Using Zorn's lemma, there is a maximal family $\mathcal{M}$ that contains $\mathcal{C}$ and has the finite intersection property. We claim that $\mathcal{M}$ is closed under finite intersections.

   Let $\mathcal{I}$ be the set of finite intersections of $\mathcal{M}$. It is clear that $\mathcal{I} \supseteq \mathcal{M}$; by maximality, it suffices to show that $\mathcal{I} \in \Omega$ in order to show that $\mathcal{I} \subseteq \mathcal{M}$, i.e. $\mathcal{I} = \mathcal{M}$.

   a. $\mathcal{I}$ contains $\mathcal{C}$ as $\mathcal{I} \supseteq \mathcal{M} \supseteq \mathcal{C}$.

   b. To check that $\mathcal{I}$ has the finite intersection property, let $\{E_k\}_{k=1}^n \subseteq \mathcal{I}$ be a finite subcollection. There exist $F_{k,\ell} \in \mathcal{M}$ such that $E_k = \bigcap_{\ell=1}^{m_k} F_{k,\ell}$ for each $k = 1, \ldots, n$ by definition of $\mathcal{I}$. Because each E_k is nonempty, the $F_{k,\ell} \in \mathcal{M}$ are nonempty as well. By the finite intersection property of $\mathcal{M}$,

   $$\bigcap_{k=1}^n E_k = \bigcap_{k=1}^n \bigcap_{\ell=1}^{m_k} F_{k,\ell} \neq \varnothing.$$

3. We claim that if a subset $F \subseteq X$ satisfies $E \cap F \neq \varnothing$ for every $E \in \mathcal{M}$, then $F \in \mathcal{M}$ as well.

   Let $\mathcal{M}' = \mathcal{M} \cup \{F\}$. As above, noting that $\mathcal{M}' \supseteq \mathcal{M} \supseteq \mathcal{C}$, it suffices to show that $\mathcal{M}' \in \Omega$ by maximality of $\mathcal{M}$. In particular, it suffices to prove $\mathcal{M}' \supseteq \{E_k\}_{k=1}^n$ has the finite intersection property, where we may assume $E_k = F$ for some k, without loss of generality $k = 1$. But then $E = \bigcap_{k=2}^n E_k \in \mathcal{M}$ by its closure under finite intersections, and E is nonempty. By hypothesis on F, we have that

   $$\bigcap_{k=1}^n E_k = F \cap \bigcap_{k=2}^n E_k = F \cap E \neq \varnothing.$$

4. For $i \in I$ and $\mathcal{F} \in \Omega$, we claim that $\pi_i(\mathcal{F}) \coloneqq \{\pi_i(E) : E \in \mathcal{F}\}$ has the finite intersection property.

   Let $\{F_k\}_{k=1}^n \subseteq \pi_i(\mathcal{F})$. There exist $\{E_k\}_{k=1}^n \subseteq \mathcal{F}$ such that $F_k = \pi_i(E_k)$ for each k, and $\bigcap_{k=1}^n E_k \neq \varnothing$ by the finite intersection property of $\mathcal{F}$. Taking $e \in \bigcap_{k=1}^n E_k$, we see that $\pi_i(e) \in \pi_i(E_k)$ for every k, so

   $$\pi_i(e) \in \bigcap_{k=1}^n \pi_i(E_k) \neq \varnothing.$$

5. For $i \in I$ and $\overline{\mathcal{M}}_i \coloneqq \{\overline{\pi_i(E)} : E \in \mathcal{M}\}$, we claim that $\bigcap_{E \in \overline{\mathcal{M}}_i} E$ is nonempty.

Let $\{\pi_i(E_k)\}_{k=1}^n$ be a finite subcollection of nonempty subsets. Per the previous step 4,

$$\bigcap_{k=1}^{n} \overline{\pi_i(E_k)} \supseteq \bigcap_{k=1}^{n} \pi_i(E_k) \neq \varnothing,$$

which shows that $\overline{\mathcal{M}}_i$ has the finite intersection property. By the compactness of X_i and Proposition 1, we see that the claim is true.

6. Using the Axiom of Choice, there exists $x_i \in \bigcap_{E \in \overline{\mathcal{M}}_i} E$ for each $i \in I$. Let $x = (x_i)_{i \in I}$ be the point in X. We claim that $x \in \overline{E}$ for every nonempty $E \in \mathcal{M}$.

   Per Lemma 2, to show that $x \in \overline{E}$, it suffices to show that every basis set B containing x has $E \cap B \neq \varnothing$. For $i \in I$ and $O_i \subseteq X_i$ open, we claim that the subbasic set $\pi_i^{-1}(O_i) \ni x$ intersects E. We have $x_i \in O_i$, and $x_i \in \overline{\pi_i(E)}$ by $E \in \mathcal{M}$. Then $O_i \cap \pi_i(E) \neq \varnothing$, and nonempty image means nonempty preimage. Now, step 3 shows that every subbasic set containing x belongs to $\mathcal{M}$; step 2 shows that basis sets containing x, finite intersections of subbasic sets, belong to $\mathcal{M}$ too. Then $B \cap E \neq \varnothing$ by the finite intersection property of $\mathcal{M}$ for $x \in B \in \mathcal{M}$.

   As every $C \in \mathcal{C}$ is closed and belongs to $\mathcal{M}$, we have shown that $\mathcal{C}$ has the finite intersection property:

$$x \in \bigcap_{C \in \mathcal{C}} \overline{C} = \bigcap_{C \in \mathcal{C}} C \neq \varnothing.$$

   Thus we are done. By Proposition 1, X is compact with respect to the product topology.

□

The proof of the converse AC $\implies$ Tychonoff's, which is much shorter than the proof above, is on the next page.

Theorem 2 (Kelley's theorem)**.**

Tychonoff's theorem implies the Axiom of Choice.

Proof of Kelley's theorem. Suppose that Tychonoff's theorem is true. Let $\{X_i\}_{i\in I}$ be any collection of nonempty sets; we wish to show that $X := \prod_{i\in I} X_i$ is nonempty.

1. First, we enlarge the X_i to equip them with suitable topologies. Let ω be a set that does not belong to any of the X_i, for instance $\omega := \bigcup_{i\in I} X_i$, which does not belong to $\bigcup_{i\in I} X_i$ by the Axiom of Regularity. Then define $Y_i := X_i \cup \{\omega\}$, to which we give the topology $\tau_i := \{\varnothing, X_i, \{\omega\}, Y_i\}$.

   It is straightforward to check that $\{\varnothing, A, A^c, S\}$ is a topology on $S \supseteq A$ in general. Moreover, each (Y_i, τ_i) is compact by the finiteness of the τ_i. Invoking Tychonoff's theorem, $Y := \prod_{i\in I} Y_i$ is compact.

2. Now, we will extract an element of X using the compactness of Y through Proposition 1. Let $\pi_i \colon Y \twoheadrightarrow Y_i$ be the canonical projection, which is continuous as Y is given the product topology. As X_i is closed in Y_i, we have that $C_i := \pi_i^{-1}(X_i)$ is closed in Y.

   We claim that the collection of closed sets $\{C_i\}_{i\in I}$ has the finite intersection property. Given finite subcollection $\{C_{i_k}\}_{k=1}^n$, we can choose finitely many points $x_{i_k} \in X_{i_k}$ and set

   $$y_i := \begin{cases} x_{i_k} & \text{if } i \in \{i_1, \dots, i_n\} \\ \omega & \text{otherwise.} \end{cases}$$

   Then the point $y = (y_i)_{i\in I}$ belongs to Y, and $y \in \pi_{i_k}^{-1}(X_{i_k}) = C_{i_k}$ for each $k = 1, \dots, n$ per definition, so $y \in \bigcap_{k=1}^n C_{i_k}$. Note that we can always choose an element out of each of *finitely* many nonempty sets; y_i is set to be ω in all other coordinates to avoid a circular argument needing AC.

3. Invoking Proposition 1, there exists some $x \in \bigcap_{i\in I} C_i$. But $x \in C_i$ means that $\pi_i(x) \in X_i$ for each $i \in I$, i.e. $x \in \prod_{i\in I} X_i$. Thus we are done.

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