Introduction to complex analysis

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This note is my own summary of Chapters 1–2.5 of *Complex Variables and Their Applications* by Brown and Churchill, 7th edition.

The complex numbers $(\mathbb{C}, +, \times, |\cdot|)$ form a complete field. $\mathbb{C} \simeq \mathbb{R}^2$ as additive groups, where each $z \in \mathbb{C}$ admits a unique representation $z = x + yi$ for $x, y \in \mathbb{R}$, which is more convenient for additive properties. In this Cartesian or rectangular form, x is the real part of z, and y is the imaginary part of z.

$$\frac{z_1}{z_2} = \frac{1}{x_2^2 + y_2^2}((x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)).$$

In the nonunique polar form $z = r(\cos\theta + i\sin\theta) \coloneqq re^{i\theta}$, $z \neq 0$, the modulus or absolute value of z is $r \coloneqq |z| = \sqrt{x^2 + y^2}$, and the argument of z is any θ which solves $\tan\theta = y/x$. The principal argument $\mathrm{Arg}(z)$ is the unique solution in $(-\pi, \pi]$, and $\arg(z)$ is the set of $2\pi n$-translates θ of $\mathrm{Arg}(z)$. Polar form is more convenient for multiplication as $\arg(\cdot)$ acts "logarithmically":

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2),$$

interpreted as $A + B = \{a + b : a \in A, b \in B\}$. De Moivre's formula follows from the exponential form:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

$\mathbb{C}$ is a *-algebra under the conjugation $z = x + yi \mapsto \overline{z} \coloneqq x - yi$, where $z\overline{z} = |z|^2$. The nth root of any $z = re^{i\theta}$ is any root of unity $\zeta^k = e^{i2\pi k/n}$, scaled by the unique positive root $\sqrt[n]{r}$ and rotated by $e^{i\theta/n}$. The principal nth root of z is $\sqrt[n]{r}e^{i\,\mathrm{Arg}\,z}$.

A neighborhood of z_0 is an open ball about z_0 of radius $\varepsilon > 0$, and a deleted neighborhood is one missing its center. Interior points lie in the interior, exterior points lie outside of the closure, and boundary points lie on the boundary. Or, an exterior point lies in a neighborhood disjoint from the set. Open sets contain none of their boundary points, and closed sets all. The punctured disk is the closed unit ball without the origin. A domain is a connected open set, like a neighborhood; a region is a domain with any number of its boundary points. Every deleted neighborhood of an accumulation point intersects the set.

This finishes a speedrun of Chapter 1.

We take the *maximal* domain of definition. Multiple-valued or multivalued functions are often of interest in complex analysis, unlike in real analysis. "Inasmuch" is in fact a word (!). To study mappings, we can examine the images of quadrants, curves, lines, rays, and regions under the transformation.

Definition 1 (Limit).

For f defined in some deleted neighborhood of z_0, the limit of $f(z)$ as z approaches z_0, if it exists, is the point w_0 where every ε-neighborhood of w_0 contains the image of some deleted δ-neighborhood of z_0. *The* limit is unique by the triangle inequality for $|\cdot|$.

We can extend Definition 1 to z_0 in the boundary of the region $\operatorname{dom} f$ by restricting z to the intersection of the deleted δ-neighborhood and the region. For example, if $f(z) = wz$, $w \in \mathbb{C}$ is defined on the open unit ball, then $\lim_{z \to i} f(z) = wi$: for any $\varepsilon > 0$, there exists $\delta = \varepsilon / |w|$ such that

$$0 < |z - i| < \delta \implies |w| \cdot |z - i| = |f(z) - wi| < \varepsilon.$$

Importantly, $z \to z_0$ approaches in an *arbitrary* manner, not from some particular direction or along some curve. For example, $\lim_{z \to 0} z/\overline{z}$ does not exist: it equals 1 when approached along the real axis and -1 along the imaginary axis, which contradicts the uniqueness of a limit.

Some regular checks show that a complex function converges at a point iff both of its real and imaginary parts converge, and complex limits still preserve addition and multiplication (and thus polynomials). To define limits at infinity, let us first define $\mathbb{C} \cup \{\infty\}$:

Definition 2 (Extended complex plane).

The extended complex plane or one-point compactification $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, where ∞ is the *point at infinity*, is the stereographic projection of the Riemann sphere.

Consider a three-dimensional sphere centered at the origin of the ordinary complex plane. Every point $z \in \mathbb{C}$ can be uniquely identified with a point P on the surface of the sphere, so that z, P, and the north pole lie on the same line. Then the south pole of the sphere corresponds to 0, while the north pole is said to correspond to ∞.

The equator of the unit sphere is precisely the unit circle; the northern hemisphere is the exterior of the unit circle, and the southern hemisphere its interior. An **ε-neighborhood of ∞** is therefore defined to be $\{z : |z| > 1/\varepsilon\}$. The expected limit results follow:

Proposition 1 (Infinite limits and limits at infinity).

Let $z_0, w_0 \in \mathbb{C}$. Then

a. $\lim_{z \to z_0} f(z) = \infty$ iff $\lim_{z \to z_0} 1/f(z) = 0$.

b. $\lim_{z \to \infty} f(z) = w_0$ iff $\lim_{z \to 0} 1/f(z) = w_0$.

Consequently, $\lim_{z\to\infty} f(z) = \infty$ iff $\lim_{z\to 0} 1/f(1/z) = 0$. In some sense, ∞ and 0 really are dual like the north pole and south pole of a sphere, furthest away from each other in modulus or in reciprocal.

The usual definition of pointwise continuity follows, and sums, products, and compositions of continuous functions are continuous per the usual proofs. In particular, polynomials are continuous. If f continuous is nonzero at a point z_0, then it is nonzero in a neighborhood of z_0: take $\varepsilon = |f(z_0)|/2$. And, a complex function is continuous iff its real and imaginary components are. It follows that continuous functions on closed bounded regions are bounded and attain an extrema: the $\mathbb{R}$-valued $|f|(z) = |f(z)|$ is continuous.

Definition 3 (Derivative).

If $\operatorname{dom} f$ contains some neighborhood of z_0, then the derivative of f at z_0, if the limit exists, is

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h}.$$

Differentiability implies continuity, but not the converse. For instance, $f(z) = |z|^2$ is continuous everywhere, but differentiable only at 0. We abbreviate $z = z_0$ and write $\Delta z = h$, $\Delta f \coloneqq f(z + \Delta z) - f(z)$.

$$\frac{\Delta f}{\Delta z} = \frac{(z + \Delta z)(\overline{z} + \overline{\Delta z}) - z\overline{z}}{\Delta z} = \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}.$$

If Δz approaches 0 along the x-axis, $\Delta f/\Delta z$ approaches $\overline{z} + z$; along the y-axis, $\Delta f/\Delta z$ approaches $\overline{z} - z$. Thus if $f'(z) = \lim_{\Delta z\to 0} \Delta f/\Delta z$, then $\overline{z} + z$ must equal $\overline{z} - z$, or $z = 0$.

The usual derivative rules — linearity, power rule, product rule, quotient rule, and chain rule — still hold for the complex derivative.

Proposition 2 (Cauchy–Riemann equations).

Suppose that $f(z) = u(x, y) + iv(x, y)$ is differentiable at $z_0 = x_0 + iy_0$. Then

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0), where the existence of the first-order partial derivatives of u and v is implicit. Moreover,

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

The contrapositive of Proposition 2 is often useful to find where f is *not* differentiable.

Proof. Keeping the notation above, we also write $\Delta z = \Delta x + i\Delta y$. Note that

$$\Delta f = [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)].$$

The derivative of f at z_0 exists by hypothesis and equals

$$f'(z_0) = \lim_{(\Delta x,\Delta y)\to(0,0)} \operatorname{Re}\left(\frac{\Delta f}{\Delta z}\right) + i \lim_{(\Delta x,\Delta y)\to(0,0)} \operatorname{Im}\left(\frac{\Delta f}{\Delta z}\right).$$

Suppose we let $(\Delta x, \Delta y)$ approach $(0,0)$ along the x-axis, so that $\Delta y = 0$. Then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0). \end{aligned}$$

Likewise, if we let $(\Delta x, \Delta y)$ approach $(0,0)$ along the y-axis, then

$$\begin{aligned} f'(z_0) &= \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\ &= -i u_y(x_0, y_0) + v_y(x_0, y_0). \end{aligned}$$

These two quantities are equal, and we are done. □

The key to the proof above: as limits allow approaching in any arbitrary manner, we chose the simplest horizontal and vertical directions to approach in, which eliminates any Δy and Δx respectively, isolating the x- and y-components of the function, which is precisely what gives access to the partial derivatives. In fact, we also have a partial, nearly full converse to Proposition 2.

Proposition 3 (Sufficient conditions for differentiability).

Suppose that $f(z) = u(x,y) + iv(x,y)$ and the first-order partial derivatives of u and v are defined in some neighborhood of the point $z_0 = x_0 + iy_0$. If the partial derivatives are continuous at (x_0, y_0) and satisfy the Cauchy–Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

at (x_0, y_0), then f is differentiable at z_0.

Proof. Keeping the notation above, let $0 < |\Delta z| < \varepsilon$ and write

$$\begin{aligned} \Delta f &= \Delta u + i \Delta v \\ &\coloneqq [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]. \end{aligned}$$

By the continuity of the first-order partial derivatives at (x_0, y_0), we can write

$$\Delta u = u_x(x_0, y_0)\Delta x + u_y(x_0, y_0)\Delta y + \varepsilon_1 \sqrt{(\Delta x)^2 + (\Delta y)^2},$$

and likewise for Δv, where $\varepsilon_1, \varepsilon_2$ tend to 0 as $(\Delta x, \Delta y) \to (0,0)$. Substituting above and applying the Cauchy–Riemann equations, we have that

$$\Delta f = u_x(x_0, y_0)(\Delta x + i\Delta y) + v_x(x_0, y_0)(i\Delta x - \Delta y) + (\varepsilon_1 + i\varepsilon_2)\sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Dividing by $\Delta z = \Delta x + i\Delta y$, we find that the limit exists and equals

$$f'(z_0) = u_x(x_0, y_0) + i v_x(x_0, y_0).$$

□

As a consequence, we see that $e^z = e^x \cos y + ie^x \sin y$ is really its own derivative, and we have another explanation for why $z \mapsto |z|^2$ is differentiable only at 0.

We will continue with deriving a polar form for the Cauchy–Riemann equations, analytic functions, and harmonic functions.

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