Selected proofs on measure theory

Alex Fu

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This note is a selection of proofs and my own solutions to exercises from *Real Analysis: Modern Techniques and Their Applications* by Gerald B. Folland, 2nd edition.

Proposition 1 (Existence of nonmeasurable sets)**.**

Let $d \geq 1$. There does not exist a function $\mu \colon \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ satisfying all three properties below.

1. *Countable additivity*. If $E_1, E_2, \ldots$ is a countable collection of (pairwise) disjoint subsets of $\mathbb{R}^d$, then

$$\mu\left(\bigsqcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n).$$

2. *Translation invariance*. If $E \subseteq \mathbb{R}^d$ and $T \colon \mathbb{R}^d \to \mathbb{R}^d$ is a linear isometry, i.e. $\det(T) = \pm 1$, then

$$\mu(E) = \mu(T(E)).$$

   In particular, translations, rotations, reflections, and any compositions thereof are linear isometries.

3. *Unit hypercube*. If Q is the unit hypercube $[0, 1)^d$ or $[0, 1]^d$, then

$$\mu(Q) = 1.$$

The following proof is a revision of 2023-01-04.

Proof. We will show that there exists a *Vitali set* $E \subseteq [0, 1)$ such that $\mu(E)$ cannot be defined for μ satisfying all three properties above. It suffices to show the case of $d = 1$ as we can consider $E \times [0, 1)^{d-1} \subseteq [0, 1) \times [0, 1)^{d-1}$ for $d \geq 1$ in general.

Define an equivalence relation $\sim$ on $[0, 1)$ such that $x \sim y$ iff $x - y \in \mathbb{Q}$. Assuming the Axiom of Choice, we may choose a representative from each equivalence class; call the set of representatives E. (For example, E could be $\{0, e - 2, \pi - 3, \ldots\}$, although this proof using AC is nonconstructive.) We also write

$$E_q := (E + q) \bmod 1 = \{(x + q) \bmod 1 : x \in E\} \subseteq [0, 1).$$

Now, we claim that the collection of all E_q, $q \in \mathbb{Q} \cap [0, 1)$ is a countable partition of $[0, 1)$.

a. $\mathbb{Q} \cap [0, 1)$ is countable, so the collection of E_q is certainly countable.

b. The union of the E_q covers $[0, 1)$, i.e. every $x \in [0, 1)$ belongs to at least one E_q: by definition of E, there exists some $y \sim x$ such that $y \in E$. Then $x = y + (x - y) \in E + (x - y) \bmod 1$, or $x \in E_{x-y}$, $x - y \in \mathbb{Q}$.

c. The E_q are disjoint, i.e. every $x \in [0, 1)$ belongs to at most one E_q: let $q, r \in \mathbb{Q}$, $q \neq r$, and suppose there is some $x \in E_q \cap E_r$. That is, $y + q = x = y' + r \bmod 1$ for some $y, y' \in E$. With $q, r \in [0, 1)$ and $q \neq r$, it is clear that $y \neq y'$. But by definition of $\sim$, we see that $y \sim y'$, which contradicts the definition of E.

Thus, if μ satisfies the conditions of Proposition 1, then

$$1 = \mu([0, 1)) = \sum_{q \in \mathbb{Q} \cap [0,1)} \mu(E_q) = \sum_{q \in \mathbb{Q} \cap [0,1)} \mu(E).$$

But the countably infinite sum of $\mu(E) \in [0, \infty]$ is either 0 or ∞, so $E \subseteq [0, 1)$ cannot be assigned a value under μ. Therefore μ cannot be defined on all of $\mathcal{P}(\mathbb{R}^1)$. □

In fact, weakening condition 1 to finite additivity still fails to produce a viable μ. A result of Banach–Tarski states that for *any* bounded open sets $O, O' \subseteq \mathbb{R}^d$, $d \geq 3$, there exist finite partitions of O and O' into an equal number of pieces $\{E_i\}_{i=1}^n$, $\{F_i\}_{i=1}^n$ such that each E_i is congruent to F_i, i.e. the image of F_i under some linear isometry (and vice versa). Thus, a sphere and two spheres could have equal "size" under a finitely additive measure.

Proposition 2 (Disjointization of a union)**.**

Let $E_1, E_2, \ldots$ be a countable collection of sets. Then there exist $F_1, F_2, \ldots$ disjoint, such that

$$\bigcup_{n=1}^{\infty} E_n = \bigsqcup_{n=1}^{\infty} F_n.$$

Proof. Consider $F_n \coloneqq E_n \setminus \bigcup_{i=1}^{n-1} E_i$, or $E_n \cap \bigcap_{i=1}^{n-1} E_i^c$, from which it is clear that the F_n are disjoint. And, by induction with base case $E_1 = F_1$,

$$\bigcup_{i=1}^{n} E_i = \bigcup_{i=1}^{n-1} E_i \sqcup \left(E_n \setminus \bigcup_{i=1}^{n-1} E_i \right) = \bigsqcup_{i=1}^{n-1} F_i \sqcup F_n = \bigsqcup_{i=1}^{n} F_i$$

for every $n \geq 1$. Taking the limit as $n \to \infty$, we are done. □

If $E_1, E_2, \ldots$ are sets in a σ-algebra in particular, then we find that every countable union in a σ-algebra can be made into a *disjoint* one. Considering the "partial unions" $E_1 \subseteq E_1 \cup E_2 \subseteq E_1 \cup E_2 \cup E_3 \subseteq \cdots$, every countable union can also be made into an *ascending* limit.

Now, some terminology: a $\boldsymbol{G_\delta}$ set is a countable intersection of open sets; a $\boldsymbol{F_\sigma}$ set a countable union of closed sets; a $\boldsymbol{G_{\delta\sigma}}$ set a countable union of G_δ sets; a $\boldsymbol{F_{\sigma\delta}}$ set a countable intersection of F_σ sets, and so forth. δ and σ stand for *Durchschnitt* and *Summe*, German for *intersection* and *union* respectively.

Proposition 3 (Generating sets of the Borel σ-algebra)**.**

The Borel σ-algebra $\mathcal{B}$ on $\mathbb{R}$ is generated by any of the following collections.

1. The open intervals $\mathcal{C}_1 = \{(a, b) : a < b\}$.

2. The closed intervals $\mathcal{C}_2 = \{[a,b] : a < b\}$.
3. The half-open half-closed intervals $\mathcal{C}_3 = \{(a,b] : a < b\}$.
4. The half-closed half-open intervals $\mathcal{C}_4 = \{[a,b) : a < b\}$.
5. The open rays $\mathcal{C}_5 = \{(-\infty, x) : x \in \mathbb{R}\}$ or $\mathcal{C}_6 = \{(x, \infty) : x \in \mathbb{R}\}$.
6. The closed rays $\mathcal{C}_7 = \{(-\infty, x] : x \in \mathbb{R}\}$ or $\mathcal{C}_8 = \{[x, \infty) : x \in \mathbb{R}\}$.

Proof. By the minimality of the generated σ-algebra (2023-01-09), it suffices to check $\mathcal{C}_i \subseteq \sigma(\mathcal{C}_j)$ to show that $\sigma(\mathcal{C}_i) \subseteq \sigma(\mathcal{C}_j)$. We will simply exhibit a useful collection of identities in this vein.

- $(a,b)^c = (-\infty, a] \cup [b, \infty)$ and $[a,b]^c = (-\infty, a) \cup (b, \infty)$; similar holds for $(a,b]^c$ and $[a,b)^c$.
- $(a,b) = (-\infty, b) \setminus (-\infty, a] = (a, \infty) \setminus (b, \infty)$ and $[a,b] = (-\infty, b] \setminus (-\infty, a) = [a, \infty] \setminus (b, \infty)$.
- $\varnothing = (x,x) = (x,x] = [x,x)$.
- $\{x\} = [x,x] = (-\infty, x] \setminus (-\infty, x)$.
- $\bigcup_{n=1}^\infty (-\infty, x_n) = (-\infty, \sup_{n\geq 1} x_n)$, and likewise for (x_n, ∞).
- $\bigcap_{n=1}^\infty (-\infty, x_n) = (-\infty, \inf_{n\geq 1} x_n)$ *or* $(-\infty, \inf_{n\geq 1} x_n]$, and likewise for (x_n, ∞).
- $[a,b] = \bigcap_{b_n \downarrow b} [a, b_n) = \bigcap_{b_n \downarrow b} [a, b_n] = \bigcap_{n=1}^\infty [a, b + \frac{1}{n}) = \bigcap_{n=1}^\infty (a - \frac{1}{n}, b]$.
- $(a,b) = \bigcup_{b_n \uparrow b} (a, b_n] = \bigcup_{b_n \uparrow b} (a, b_n) = \bigcup_{n=1}^\infty (a, b - 2^{-n}] = \bigcup_{n=1}^\infty (a + 2^{-n}, b)$.

□

For an informal summary, the complement of a closed endpoint is an "open endpoint," and vice versa; to create a closed endpoint, approach from the outside, or "reduce down to a point" through intersections; to create an open endpoint, approach from the inside, or "build right up to the edge" through unions.

As noted before, the denseness of the rationals in the reals is also helpful when converting seemingly uncountable situations to countable ones, e.g. "for all $\varepsilon > 0$" to "as $\frac{1}{n} \downarrow 0$." Here, we can also introduce countability:

Proposition 4 (Countable generating set of the Borel σ-algebra)**.**

The Borel σ-algebra on $\mathbb{R}^d$ is countably generated.

Proof. We claim that $\mathcal{B}$ is generated by the set of open intervals with rational endpoints $\{(a,b) : a, b \in \mathbb{Q}, a < b\}$. Indeed, every $x \in \mathbb{R}$ admits a (monotone) sequence of rationals converging to it, without loss of generality $q_n \uparrow x$. Then $[x, \infty) = \bigcap_{n=1}^\infty (q_n, \infty)$, or use any of the variations from Proposition 3.

By later results on product σ-algebras, we see that the Borel σ-algebra on $\mathbb{R}^d$ is generated by so-called *measurable rectangles*, or products of one-dimensional Borel sets. Thus the product of generating collections itself generates the product σ-algebra. As the product of (countably many) countable sets is countable, we are done. □

In the interest of time, we collect some miscellaneous results below in no particular order.

Proposition 5 (Cardinality gap of σ-algebras)**.**

Let $\mathcal{F}$ be an infinite σ-algebra. Then $\mathcal{F}$ is uncountable.

Proof. By hypothesis, there exist $E_1, E_2, \ldots \in \mathcal{F}$. By disjointization (Proposition 2), there is a countable sequence of disjoint sets $F_1, F_2, \ldots \in \mathcal{F}$. Now, $2^{\mathbb{N}}$ is uncountable, so

$$\left\{ \bigcup_{i \in I} F_i : I \subseteq \mathbb{N} \right\} \subseteq \mathcal{F},$$

which is in bijection with $2^{\mathbb{N}} = \{I : I \subseteq \mathbb{N}\}$ by the disjointness of the F_n, is also uncountable. □

Proposition 6 (Monotone class lemma)**.**

An algebra $\mathcal{A}$ is a σ-algebra iff it is closed under increasing unions (and decreasing intersections).

Proof. One direction is trivial. For the other direction, if $E_1, E_2, \ldots \in \mathcal{A}$, consider $\bigcup_{i=1}^{n} E_i \uparrow \bigcup_{i=1}^{\infty} E_i \in \mathcal{A}$. □

Proposition 7 (σ-algebras are the union of all countably-generated sub-σ-algebras)**.**

If $\mathcal{F}$ is a σ-algebra generated by $\mathcal{C}$, then

$$\mathcal{F} = \bigcup \{\sigma(\mathcal{S}) : \mathcal{S} \subseteq \mathcal{C},\, \mathcal{S} \text{ countable}\}.$$

Proof. Let $\mathcal{U}$ be the union above. It is clear that $\mathcal{C} \subseteq \mathcal{U} \subseteq \mathcal{F}$, so we check that $\mathcal{U}$ is a σ-algebra.

1. If $A \in \mathcal{U}$, then it is contained in some $\sigma(\mathcal{S})$ which also contains A^c. Thus $A^c \in \mathcal{U}$.
2. If $A_1, A_2, \ldots \in \mathcal{U}$, there exist $\mathcal{S}_1, \mathcal{S}_2, \ldots \subseteq \mathcal{C}$ such that $A_n \in \sigma(\mathcal{S}_n)$. Then $\bigcup_{n=1}^{\infty} A_n \in \sigma(\bigcup_{n=1}^{\infty} \mathcal{S}_n) \subseteq \mathcal{U}$.

□

Proposition 8 (Rings and σ-rings)**.**

A ring is a family closed under finite unions and set differences, and a σ-ring is closed under countable unions.

a. (σ-)rings are closed under (countably in)finite intersections.

b. A (σ-)ring $\mathcal{R}$ is a (σ-)algebra iff $X \in \mathcal{R}$.

c. If $\mathcal{R}$ is a (σ-)ring, then $\mathcal{C} = \{E \subseteq X : E \in \mathcal{R} \text{ or } E^c \in \mathcal{R}\}$ is a (σ-)algebra.

d. If $\mathcal{R}$ is a (σ-)ring, then $\mathcal{I} = \{E \subseteq X : E \cap F \in \mathcal{R} \text{ for all } F \in \mathcal{R}\}$ is a (σ-)algebra.

Proof. I'm sure there are some interesting connections beyond name between a ring of sets, Boolean algebra, set operations, and an algebraic ring. Here is some more filler text to make this page break cleaner.

a. If $E_1, \dots, E_n \in \mathcal{R}$, let $F = \bigcup_{i=1}^n E_i \in \mathcal{R}$. Then $\bigcap_{i=1}^n E_i = F \setminus \bigcup_{i=1}^n (F \setminus E_i)$ by De Morgan's laws. The same proof holds for $n = \infty$ in a σ-ring $\mathcal{R}$.

b. If $X \in \mathcal{R}$, then $\mathcal{R}$ is closed under complements, which makes it a (σ-)algebra. If (σ-)ring $\mathcal{R}$ is a (σ-)algebra, then because $\varnothing = E \setminus E \in \mathcal{R}$ always, we have $\varnothing^c = X \in \mathcal{R}$.

c. $\mathcal{C}$ is clearly closed under complements. Moreover, if $F_1, \dots, F_n \in \mathcal{C}$, then $\bigcup_{i=1}^n F_i \in \mathcal{R} \subseteq \mathcal{C}$: s'pose $F_i^c \in \mathcal{R}$ for $i \in I \subseteq [n]$. Then $\bigcup_{i \notin I} F_i \cup X \setminus \bigcap_{i \in I} F_i^c \in \mathcal{R}$. Note that $X = \varnothing^c \in \mathcal{C}$, which makes $\mathcal{C}$ a (σ-)algebra by part b.

d. $X \in \mathcal{I}$, so it suffices to show that $\mathcal{I}$ is a (σ-)ring. If $E \in \mathcal{I}$, then for all $F \in \mathcal{R}$, $E^c \cap F = F \setminus (E \cap F) \in \mathcal{R}$, i.e. $E^c \in \mathcal{I}$ as well. If $E_1, \dots, E_n \in \mathcal{I}$, then $(\bigcup_{i=1}^n E_i) \cap F = \bigcup_{i=1}^n (E_i \cap F) \in \mathcal{R}$, i.e. $\bigcup_{i=1}^n E_i \in \mathcal{I}$.

□

Lastly, we made note of the lack of an "explicit" characterization of $\mathcal{F} = \sigma(\mathcal{C})$ given $\mathcal{C}$: it is not enough to iterate the operations of countable union and intersection countably many times. Now, using a few more tools from set theory, we can give a slightly clearer answer.

With $\mathcal{C}' = \mathcal{C} \cup \{C^c : C \in \mathcal{C}\}$, we can assume without loss of generality that $\mathcal{C}_1 = \mathcal{C}$ is closed under complements. Let $\mathcal{C}_2 = \mathcal{M}^1(\mathcal{C}_1)$ be the set of all countable unions and intersections in $\mathcal{C}_1$, or equivalently all countable unions and complements thereof, so $\mathcal{C}_2$ is closed under complements as well. Inducting, let $\mathcal{C}_\omega = \bigcup_{n=1}^\infty \mathcal{C}_n$. $\mathcal{C}_\omega$ is closed under complements, but if $E_n \in \mathcal{C}_n \setminus \mathcal{C}_{n-1}$, $\bigcup_{n=1}^\infty E_n$ may not belong to $\mathcal{C}_\omega$. We must go further.

Define $\mathcal{C}_\alpha$ for every countable ordinal α by transfinite induction: if α is the successor of β, then let $\mathcal{C}_\alpha$ be the set of all countable unions and intersections in $\mathcal{C}_\alpha$; otherwise, let $\mathcal{C}_\alpha = \bigcup_{\beta \in \alpha} \mathcal{C}_\beta$. Letting Ω be the set of countable ordinals, we have the following result.

Proposition 9 (Constructing a generated σ-algebra)**.**

$\sigma(\mathcal{C}) = \bigcup_{\alpha \in \Omega} \mathcal{C}_\alpha$.

Proof. $\mathcal{C}_\alpha \subseteq \sigma(\mathcal{C})$ for all $\alpha \in \Omega$ by transfinite induction, hence $\bigcup_{\alpha \in \Omega} \mathcal{C}_\alpha \subseteq \sigma(\mathcal{C})$. For the other direction, every countable subset A of Ω has an upper bound: $\bigcup_{\alpha \in A} \operatorname{seg}(\alpha)$ is countable, and thus a proper subset of Ω, so there exists $\beta \in \Omega$ such that $\bigcup_{\alpha \in A} \operatorname{seg}(\alpha) = \operatorname{seg}(\beta)$. If $E_n \in \mathcal{C}_{\alpha_n}$ for all $n \geq 1$ and $\beta := \sup_{n \geq 1} \alpha_n$, then $E_n \in \mathcal{C}_\beta$ for all n, and $\bigcup_{n=1}^\infty E_n \in \mathcal{C}_{\beta+}$. Thus $\bigcup_{\alpha \in \Omega} \mathcal{C}_\alpha$ is a σ-algebra and equals $\sigma(\mathcal{C})$ by minimality. □

We will continue with Chapter 1.3 of Folland.

■