Carathéodory's extension theorem

Alex Fu

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We began the construction of the Lebesgue measure on $\mathbb{R}$ in *2023-01-06*, extending the length premeasure to an algebra and defining the Borel σ-algebra. Recall the following theorem we have yet to prove:

Theorem 1 (Carathéodory's extension theorem)**.**

> Let μ be a premeasure on an algebra $\mathcal{A}$. Then there exists a measure $\bar{\mu}$ on Σ, the σ-algebra generated by $\mathcal{A}$, such that $\bar{\mu} \upharpoonright \mathcal{A} \equiv \mu$; that is, $\bar{\mu}$ is an extension of μ. If μ is also σ-finite, then $\bar{\mu}$ is unique.

Here, we will make good on the promise made in *2023-01-06*. First, we will give two proofs, using Dynkin's π-λ lemma and the monotone class lemma, of the uniqueness result in Theorem 1, which we started in *2023-01-07*. Then, we will show the existence part of Theorem 1 using outer measures, and briefly consider the completion of a σ-algebra. We leave product σ-algebras and product measures to another time, as a continuation of both *2023-01-06* and *2023-01-14*.

1 Uniqueness of σ-finite extensions

As noted, this section continues from *2023-01-07*.

Proposition 1 (σ-finite measures are determined on any generating π-system)**.**

> Let μ_1, μ_2 be σ-finite measures on Σ. If Σ is generated by some π-system Π such that $\mu_1(A) = \mu_2(A)$ for all $A \in \Pi$, then $\mu_1 = \mu_2$ (on all of Σ).

The hypotheses above are in some sense necessary. σ-finiteness ensures that μ_1, μ_2 are "small enough" as to not be infinite on all of Π, and thus not uniquely extendable from Π. For example, let $\mathcal{S}$ be the usual semialgebra of half-open half-closed intervals of $\mathbb{R}$, which is a π-system, let μ_1 be the counting measure, and let $\mu_2 = 2\mu_1$. Then $\mu_1(A) = \mu_2(A) = \infty$ for all nonempty $A \in \mathcal{S}$, but clearly μ_1 and μ_2 do not agree on all of Σ, e.g. $\mu_1(\{0\}) = 1 \neq 2 = \mu_2(\{0\})$. Here, μ_1, μ_2 are not σ-finite as $\mathbb{R}$ is uncountable.

There are also many examples of μ_1, μ_2 that agree on a mere generating *set* $\mathcal{C}$ but not on $\sigma(\mathcal{C})$. A simple example is given by $\Omega = \{1, 2, 3, 4\}$,

$$\mathcal{C} = \{\varnothing, \{1,2\}, \{3,4\}, \{1,3\}, \{2,4\}, \Omega\},$$

$\mu_1(\{*\}) = \frac{1}{4}$ on all singletons, and $\mu_2(\{1\}) = \mu_2(\{4\}) = \frac{1}{2}$. Then $\mu_1 \equiv \mu_2$ on $\mathcal{C}$, but not on $\sigma(\mathcal{C}) = 2^\Omega$ when extended using finite additivity.

Proposition 2 (Properties of a measure).

a. *Monotonicity.* If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

b. *Set difference.* If $\mu(A) < \infty$ and $A \subseteq B$, then $\mu(B \setminus A) = \mu(B) - \mu(A)$.

c. *Continuity from below.* If $A_1 \subseteq A_2 \subseteq \cdots$ has ascending limit $A_n \uparrow \bigcup_{i=1}^\infty A_i$, then

$$\mu(A_n) \uparrow \mu\left(\bigcup_{i=1}^\infty A_i\right).$$

d. *Continuity from above.* If $A_1 \supseteq A_2 \supseteq \cdots$ has descending limit $A_n \downarrow \bigcap_{i=1}^\infty A_i$, and $\mu(A_i) < \infty$ for some $i \geq 1$ (without loss of generality $i = 1$), then

$$\mu(A_n) \downarrow \mu\left(\bigcap_{i=1}^\infty A_i\right).$$

Proof. If all we have about μ is countable additivity, then we should form partitions and disjoint unions to use countable additivity.

a. Note that $B = A \sqcup (B \setminus A)$, optionally $\sqcup \varnothing \sqcup \cdots$. Then $\mu(A) + \mu(B \setminus A) = \mu(B)$, $\mu(B \setminus A) \geq 0$.

b. From part a, we can subtract $\mu(A)$ from both sides if $\mu(A) < \infty$; otherwise, $\mu(B) - \mu(A) = \infty - \infty$ is indeterminate.

c. Consider the disjointization $B_n \coloneqq A_n \setminus A_{n-1}$, such that $A_n = \bigcup_{i=1}^n A_i = \bigsqcup_{i=1}^n B_i$ and

$$\mu(A_n) = \sum_{i=1}^n \mu(B_i) \uparrow \sum_{i=1}^\infty \mu(B_i) = \mu\left(\bigsqcup_{i=1}^\infty B_i\right) = \mu\left(\bigcup_{i=1}^\infty A_i\right).$$

d. Consider the ascending sequence of sets $B_n \coloneqq A_1 \setminus A_n$. From part c, we have that

$$\mu(A_1) - \mu(A_n) = \mu(A_1 \setminus A_n) \uparrow \mu\left(A_1 \setminus \bigcap_{i=1}^\infty A_i\right) = \mu(A_1) - \mu\left(\bigcap_{i=1}^\infty A_i\right).$$

If $\mu(A_1) < \infty$, then all of the measures above are finite, and we get $\mu(A_n) \downarrow \mu(\bigcap_{i=1}^{\infty} A_i)$.

$\square$

Proposition 3 (σ-finite measures agree on a λ-system)**.**

Let μ_1, μ_2 be σ-finite measures on Σ, $\mu_1(\Omega) = \mu_2(\Omega)$. Then $\{A \in \Sigma : \mu_1(A) = \mu_2(A)\}$ is a λ-system.

Proof. Proposition 2 in fact motivates the definition of a λ-system. Let $\Lambda := \{A \in \Sigma : \mu_1(A) = \mu_2(A)\}$, and suppose μ_1, μ_2 are finite without loss of generality.

1. *Nonempty.* $\Omega \in \Lambda$ by hypothesis.
2. *Closure under set difference.* If $A, B \in \Lambda$, $A \subseteq B$, then $B \setminus A \in \Lambda$ by $\mu_1(B \setminus A) = \mu_1(B) - \mu_1(A) = \mu_2(B) - \mu_2(A) = \mu_2(B \setminus A)$.
3. *Closure under increasing unions.* If $A_1, A_2, \ldots \in \Lambda$, then $A = \bigcup_{i=1}^{\infty} A_i \in \Lambda$ by the uniqueness of a limit of real numbers: $\mu_1(A) = \lim_{n\to\infty} \mu_1(A_n) = \lim_{n\to\infty} \mu_2(A_n) = \mu_2(A)$.

The equivalent definition of a λ-system — nonempty, closed under complements, and closed under countable disjoint unions — is similarly straightforward to check.

Now, if μ_1, μ_2 are σ-finite, we write $\Omega = \bigsqcup_{i=1}^{\infty} \Omega_i$. If $A, B \in \Lambda$, $A \subseteq B$, then $C = B \setminus A \in \Lambda$ by

$$\mu_1(C) = \sum_{i=1}^{\infty} \mu_1(C \cap \Omega_i) = \sum_{i=1}^{\infty} \mu_1((B \cap \Omega_i) \setminus (A \cap \Omega_i)) = \sum_{i=1}^{\infty} \mu_2((B \cap \Omega_i) \setminus (A \cap \Omega_i)) = \mu_2(C),$$

as μ_1, μ_2 are finite measures on each of the Ω_i. Likewise, Λ remains closed under increasing unions by considering the countable partition of A_n, i.e.

$$\mu_1(A) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_1(A_n \cap \Omega_i) = \lim_{n\to\infty} \sum_{i=1}^{\infty} \mu_2(A_n \cap \Omega_i) = \mu_2(A).$$

$\square$

Proposition 4 (π-λ)**.**

A family is a σ-algebra iff it is both a π-system and λ-system.

Proof. The forward direction is trivial, so let Σ be a π-system and λ-system.

1. *Nonempty.* $\Omega \in \Sigma$ per the definition of a λ-system.
2. *Closure under complements.* This is given by definition, or by the fact that $\Omega \in \Sigma$, so $A \in \Sigma \implies \Omega \setminus A \in \Sigma$.

3. *Closure under countable unions*. If $A_1, A_2, \ldots \in \Sigma$, then $B_n \coloneqq \bigcup_{i=1}^n A_i = (\bigcap_{i=1}^n A_i^c)^c \in \Sigma$ by the closure of a π-system under finite intersections and closure under complements shown above. Then $B_1 \subseteq B_2 \subseteq \cdots$, and $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i \in \Sigma$ by closure under increasing unions.

□

Theorem 2 (Dynkin's π-λ lemma).

Let Π be a π-system and Λ a λ-system. If $\Pi \subseteq \Lambda$, then $\sigma(\Pi) \subseteq \Lambda$.

Proof. Let $\mathcal{L} = \lambda(\Pi)$ be the λ-system generated by Π, so that $\Pi \subseteq \mathcal{L} \subseteq \Lambda$. Let us prove that $\mathcal{L}$ is a σ-algebra, which implies $\sigma(\Pi) \subseteq \mathcal{L} \subseteq \Lambda$. By Proposition 4, it suffices to show that $\mathcal{L}$ is a π-system.

We are given that for all $A \in \Pi, B \in \Pi$, $A \cap B \in \mathcal{L}$, and we want to show this for all $A \in \mathcal{L}, B \in \mathcal{L}$. Let us use the strategy of *successive upgrades* from *2023-01-09*: we take an intermediate step $(\Pi, \Pi) \to (\Pi, \mathcal{L}) \to (\mathcal{L}, \mathcal{L})$.

i. Claim: For all $A \in \Pi, B \in \mathcal{L}$, $A \cap B \in \mathcal{L}$. We use the common strategy of leveraging *minimality*: for fixed $A \in \Pi$, let $\mathcal{L}_A \coloneqq \{B \in \mathcal{L} : B \cap A \in \mathcal{L}\} \subseteq \mathcal{L}$. We want $\mathcal{L} \subseteq \mathcal{L}_A$, so let us check that $\mathcal{L}_A$ is a λ-system, so that $\Pi \subseteq \mathcal{L}_A$ implies $\lambda(\Pi) = \mathcal{L} \subseteq \mathcal{L}_A$.

   1. *Nonempty*. $\Omega \in \mathcal{L}$ and $\Omega \cap A = A \in \Pi \subseteq \mathcal{L}$, so $\Omega \in \mathcal{L}_A$.

   2. *Closure under set difference*. If $C, D \in \mathcal{L}_A$, $C \subseteq D$, then $C \cap A, D \cap A \in \mathcal{L}$, $C \cap A \subseteq D \cap A$. By definition of $\mathcal{L}$ as a λ-system, $(D \cap A) \setminus (C \cap A) = (D \setminus C) \cap A \in \mathcal{L}$, i.e. $D \setminus C \in \mathcal{L}_A$.

   3. *Closure under increasing unions*. If $C_1, C_2, \ldots \in \mathcal{L}_A$, then $D_n \coloneqq C_n \cap A \in \mathcal{L}$, $D_1 \subseteq D_2 \subseteq \cdots$. By definition of $\mathcal{L}$ as a λ-system, $\bigcup_{i=1}^\infty D_i = (\bigcup_{i=1}^\infty C_i) \cap A \in \mathcal{L}$, i.e. $\bigcup_{i=1}^\infty C_i \in \mathcal{L}_A$.

   We have shown that $\mathcal{L}_A = \{B \in \mathcal{L} : B \cap A \in \mathcal{L}\}$ is a λ-system, so $\mathcal{L}_A \supseteq \mathcal{L} = \lambda(\Pi)$ for any $A \in \Pi$, which proves the claim.

ii. Claim: For all $A \in \mathcal{L}, B \in \mathcal{L}$, $A \cap B \in \mathcal{L}$. We proceed as above, only now fixing $B \in \mathcal{L}$ and defining $\mathcal{L}_B = \{A \in \mathcal{L} : A \cap B \in \mathcal{L}\}$. The claim is that $\mathcal{L} \subseteq \mathcal{L}_B$, but as $\Pi \subseteq \mathcal{L}_B$ per the previous step, it suffices to show that $\mathcal{L}_B$ is a λ-system, in which case $\lambda(\Pi) = \mathcal{L} \subseteq \mathcal{L}_B$ by minimality.

   1. *Nonempty*. $\Omega \in \mathcal{L}_B$ by the same argument as above.

   2. *Closure under set difference*. If $C, D \in \mathcal{L}_B$, $C \subseteq D$, then $D \setminus C \in \mathcal{L}_B$ by the same argument as above.

   3. *Closure under increasing unions*. Follows from the same argument.

Therefore $\mathcal{L} = \lambda(\Pi)$ is a π-system (and a σ-algebra as well). As $\Pi \subseteq \Lambda$, we have that $\sigma(\Pi) = \lambda(\Pi) \subseteq \Lambda$, and we are done. □

Now, this slightly esoteric result in the Boolean algebra of sets gives a proof of Proposition 1.

Proof. By Proposition 3, $\Lambda := \{A \in \Sigma : \mu_1(A) = \mu_2(A)\}$ is a λ-system, where μ_1, μ_2 are σ-finite as given. By hypothesis, Λ contains a π-system Π that generates Σ. By Dynkin's π-λ lemma, $\Sigma \subseteq \Lambda(\subseteq \Sigma)$, i.e. $\Sigma = \Lambda$. In other words, for all $A \in \Sigma$, we have that $\mu_1(A) = \mu_2(A)$. $\square$

Dynkin's π-λ lemma is frequently also referred to as the "monotone class theorem," a distinct but related result which gives an alternative proof of Proposition 1.

Theorem 3 (Monotone class lemma).

A **monotone class** is a family closed under monotone limits, i.e. increasing unions and decreasing intersections. If $\mathcal{A}$ is an algebra and $M(\mathcal{A})$ the monotone class generated by $\mathcal{A}$, then $M(\mathcal{A}) = \sigma(\mathcal{A})$.

Proof. σ-algebras are monotone classes, so $M(\mathcal{A}) \subseteq \sigma(\mathcal{A})$. In the other direction, let us first reduce our claim to simpler cases. We write $\mathcal{M} := M(\mathcal{A})$.

i. It suffices to show that $\mathcal{M}$ is a σ-algebra by minimality. Moreover, it suffices to show that $\mathcal{M}$ is an algebra: $\mathcal{M}$ is already closed under increasing unions. If $A_1, A_2, \ldots \in \mathcal{M}$, and $B_n = \bigcup_{i=1}^n A_i \in \mathcal{M}$, then $\bigcup_{i=1}^\infty A_i = \bigcup_{i=1}^\infty B_i \in \mathcal{M}$.

ii. Fix $E \in \mathcal{M}$, and define $\mathcal{M}_E := \{F \in \mathcal{M} : (F \setminus E), (E \setminus F), (E \cap F) \in \mathcal{M}\}$. Note that if A were chosen from $\mathcal{A}$, then $\mathcal{A} \subseteq \mathcal{M}_A$, as $\mathcal{A}$ is already an algebra by hypothesis, and $\mathcal{A} \subseteq \mathcal{M}$. We claim that $\mathcal{M}_A$ is a monotone class. Given this claim, $\mathcal{M} \subseteq \mathcal{M}_A(\subseteq \mathcal{M})$ by minimality. That is, $E \in \mathcal{M}_A$ for every $E \in \mathcal{M}, A \in \mathcal{A}$, and by symmetry $A \in \mathcal{M}_E$ as well. In other words, $\mathcal{A} \subseteq \mathcal{M}_E$. Repeating the argument, we see that $\mathcal{M} = \mathcal{M}_E$ for any $E \in \mathcal{M}$. Because $\Omega \in \mathcal{A} \subseteq \mathcal{M}$, $\mathcal{M}$ is closed under complements and thus an algebra.

iii. We check the claim that $\mathcal{M}_E = \{F \in \mathcal{M} : (F \setminus E), (E \setminus F), (E \cap F) \in \mathcal{M}\}$ is a monotone class for any $E \in \mathcal{M}$. Let $F_1, F_2, \ldots \in \mathcal{M}_E$.

   1. *Closure under increasing unions.* If $F_1 \subseteq F_2 \subseteq \cdots$ and $F = \bigcup_{i=1}^\infty F_i$, then $F_i \setminus E$, $E \setminus F_i$, and $E \cap F_i$ all belong to $\mathcal{M}$. As $\mathcal{M}$ is itself a monotone class, we have $F \setminus E = \bigcup_{i=1}^\infty (F_i \setminus E)$; $E \setminus F = \bigcap_{i=1}^\infty (E \setminus F_i)$; and $E \cap F = \bigcup_{i=1}^\infty (E \cap F_i)$ all in $\mathcal{M}$ as well, i.e. $F \in \mathcal{M}_E$.

   2. *Closure under decreasing intersections.* If $F_1 \supseteq F_2 \supseteq \cdots$ and $F = \bigcap_{i=1}^\infty F_i$, then observe that $F \setminus E = \bigcap_{i=1}^\infty (F_i \setminus E)$; $E \setminus F = \bigcup_{i=1}^\infty (E \setminus F_i)$; and $E \cap F = \bigcap_{i=1}^\infty (E \cap F_i)$. Repeating the argument as above, we find that $F \in \mathcal{M}_E$.

   Therefore $\mathcal{M}_E$ is a monotone class, and $\mathcal{M}_E = \mathcal{M}$, which shows that $\mathcal{M}$ is an algebra.

$\square$

Second proof of Proposition 1. By hypothesis, $\mathcal{M} := \{A \in \Sigma : \mu_1(A) = \mu_2(A)\}$ contains some algebra $\mathcal{A}$ that generates Σ. By Proposition 2, $\mathcal{M}$ is a monotone class. By the monotone class lemma, $\mathcal{M} = \Sigma$. $\square$

The proof above is rather limited to the case of a measure already determined on an algebra, e.g. in the construction of the Lebesgue measure. The proof using Dynkin's π-λ lemma tends to be more popular, as a π-system is more versatile: all algebras and semialgebras are π-systems. For applications like cumulative distribution functions and checking independence, only requiring closure under finite intersection is much more convenient than having to specify a measure on an entire generating algebra. This may be in part why Dynkin's π-λ lemma also takes the name of "monotone class lemma."

2 Lemmas on premeasures and outer measures

True to the spirit of analysis, just as we approximated areas as Riemann sums of simpler rectangles, we wish to be able to approximate the measure of a general Borel set from within and out. In particular, if we cover a region by small boxes, the more finely we trace out the region, the more closely we approach its measure. This gives us the idea of regularity and *outer* measures.

I am unfortunately out of time today.

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3 Proof of Carathéodory's extension theorem

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4 The completion of a σ-algebra

Definition 1 (Completion).

Let μ be a measure on $\mathcal{F}$, and let $\mathcal{N}$ be the collection of all subsets of μ-null sets in $\mathcal{F}$, i.e.

$$\mathcal{N} := \{N^* : N^* \subseteq N \text{ for some } N \in \mathcal{F},\, \mu(N) = 0\}\,.$$

Then the **completion** of $\mathcal{F}$ is $\bar{\mathcal{F}} = \bar{\mathcal{F}}^\mu := \sigma(\mathcal{F}, \mathcal{N})$. A σ-algebra $\mathcal{F}$ is *complete* with respect to μ if it contains all *negligible sets* (subsets of μ-null sets), or equivalently if $\mathcal{F}$ is its own completion.

Completions and complete σ-algebras are more convenient to work with, because if $N^* \subseteq N$ and $\mu(N) = 0$, then we would like to say $\mu(N^*) = 0$ by monotonicity. The only obstacle is that N^* may not be measurable, even if the value we would assign as its measure is clear. In some sense, anything that should be measurable is indeed measurable in a complete space.

We also have a much simpler and more explicit characterization of $\bar{\mathcal{F}} = \sigma(\mathcal{F} \cup \mathcal{N})$:

Proposition 5 (The completion of a σ-algebra).

$\bar{\mathcal{F}} = \{A \cup N^* : A \in \mathcal{F} \text{ and } N^* \subseteq N \text{ for some } N \in \mathcal{F},\, \mu(N) = 0\}$.

Proof. Let $\mathcal{G}$ denote the set of unions. $\mathcal{G}$ contains $\mathcal{F}$ and $\mathcal{N}$, so it suffices to show that $\mathcal{G}$ is a σ-algebra, from which $\bar{\mathcal{F}} = \mathcal{G}$ follows by minimality.

1. *Nonempty.* $\mathcal{F} \subseteq \mathcal{G}$.

2. *Closure under complements.* For any $B = A \cup N^* \in \mathcal{G}$, there exists some μ-null $N \in \mathcal{F}$, $N^* \subseteq N$. Then $(A \cup N^*)^c = A^c \cap (N^*)^c = A^c \cap ((\Omega \setminus N) \cup (N \setminus N^*))$, and distributing the A^c-intersection, $= (A^c \cap N^c) \cup (A^c \cap (N \setminus N^*))$. Note that $(A^c \cap N^c) \in \mathcal{F}$ and the other set $\subseteq N$, so $(A \cup N^*)^c$ belongs to $\mathcal{G}$ as well.

3. *Closure under countable unions.* For $B_n = A_n \cup N_n^* \in \mathcal{G}$, with corresponding null sets $N_n \in \mathcal{F}$, we have $\bigcup_{i=1}^\infty B_i = (\bigcup_{i=1}^\infty A_i) \cup (\bigcup_{i=1}^\infty N_i^*) := A \cup N^*$, where $N^* \subseteq \bigcup_{i=1}^\infty N_i$, which is a μ-null set by countable subadditivity: $\mu(\bigcup_{i=1}^\infty N_i) \leq \sum_{i=1}^\infty \mu(N_i) = 0$. With $A \in \mathcal{F}$, we are done.

□

The **Lebesgue σ-algebra** is the completion of the Borel σ-algebra with respect to the Lebesgue measure. It is usually more convenient for the domain σ-algebra of a measurable function to be complete, as we find in the following result.

Proposition 6 (Every measurable function has a version on the completion).

Let f be a $\mathcal{F}/\mathcal{B}$-measurable function. Then there exists a $\bar{\mathcal{F}}/\mathcal{B}$-measurable function g such that $f = g$ μ-a.s. The converse holds as well: every $\bar{\mathcal{F}}/\mathcal{B}$-measurable function is μ-a.s. equal to a $\mathcal{F}/\mathcal{B}$-measurable function (called a *version* of the function).

Proof. Left to another time, along with the inner and outer regularity of the Lebesgue measure. □

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