Introduction to complex analysis continued

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We finish the summary of Chapter 2 of *Complex Variables and Their Applications* by Brown and Churchill, 7th edition, which we started in *2023-01-13*.

Proposition 1 (Polar form of Cauchy–Riemann equations)**.**

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be defined in some neighborhood of some nonzero point $z_0 = r_0 e^{i\theta_0}$, and suppose that the first-order partial derivatives of u, v with respect to r and θ exist in said neighborhood. If these partial derivatives are continuous at (r_0, θ_0) and satisfy the polar Cauchy–Riemann equations

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

at the point (r_0, θ_0), then f is differentiable at z_0.

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

when evaluated at (r_0, θ_0).

Proof. By the chain rule, using the relations $x = r\cos\theta$ and $y = r\sin\theta$, we have that

$$u_r = u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

and likewise for v_r, v_θ. If the partial derivatives with respect to x, y satisfy $u_x = v_y$, $u_y = -v_x$, then we get

$$v_r = -u_y \cos\theta + u_x \sin\theta, \quad v_\theta = u_y r\sin\theta + u_x r\cos\theta.$$

From this, it follows that $ru_r = v_\theta$ and $u_\theta = -rv_r$. Conversely, if the polar Cauchy–Riemann equations hold, then by the chain rule again,

$$u_x = u_r \cdot r_x + u_\theta \cdot \theta_x = u_r \cos\theta - u_\theta \frac{\sin\theta}{r}, \quad u_y = u_r \cdot r_y + u_\theta \cdot \theta_y = u_r \sin\theta + u_\theta \frac{\cos\theta}{r},$$

and likewise for v_x, v_y. Note that

$$\frac{\partial\theta}{\partial x} = \frac{1}{r} \cdot \frac{\mathrm{d}\theta}{\mathrm{d}\cos\theta} = \frac{1}{r} \cdot \frac{-1}{\sqrt{1-\cos^2\theta}} = -\frac{1}{r\sin\theta}, \quad \frac{\partial\theta}{\partial y} = \frac{\partial \arcsin(y/r)}{\partial y} = \frac{1}{\sqrt{r^2 - y^2}} = \frac{1}{r\cos\theta}.$$

In other words, the Cartesian Cauchy–Riemann equations are satisfied at $z_0 = (x_0, y_0)$ iff the polar Cauchy–Riemann equations are satisfied at $z_0 = (r_0, \theta_0)$. If we suppose that $v_\theta = ru_r$, $v_r = (-1/r)u_\theta$, then

$$v_x = \frac{-1}{r} u_\theta \cos\theta - ru_r \frac{\sin\theta}{r} = -\left(u_\theta \frac{\cos\theta}{r} + u_r \sin\theta\right) = -u_y$$

$$v_y = \frac{-1}{r} u_\theta \sin\theta + ru_r \frac{\cos\theta}{r} = u_x.$$

Thus the two forms of the Cauchy–Riemann equations are equivalent. □

Corollary 1.

With everything the same as in Proposition 1, $f'(z_0) = \frac{-i}{z_0}(u_\theta + iv_\theta)$.

Proof. Given that $f'(z_0) = u_x + iv_x$, everything evaluated at $z_0 = (x_0, y_0)$, we find that

$$\begin{aligned} f'(z_0) &= \left(u_r \cos\theta - u_\theta \frac{\sin\theta}{r}\right) + i\left(v_r \cos\theta - v_\theta \frac{\sin\theta}{r}\right) \\ &= (u_r + iv_r)\cos\theta - (u_r + iv_r)\sin\theta \\ &= e^{-i\theta}(u_r + iv_r). \end{aligned}$$

Now, given $f'(z_0) = e^{-i\theta}(u_r + iv_r)$, by the polar Cauchy–Riemann equations,

$$f'(z_0) = e^{-i\theta}\left(\frac{v_\theta - iu_\theta}{r}\right) = \frac{1}{re^{i\theta}} \frac{u_\theta + iv_\theta}{i} = \frac{-i}{z_0}(u_\theta + iv_\theta).$$

□

Proposition 2 (Complex form of Cauchy–Riemann equations)**.**

If $f(z) = u(x, y) + iv(x, y)$ satisfies the Cartesian Cauchy–Riemann equations, then $\partial f/\partial\overline{z} = 0$, where

$$\frac{\partial}{\partial\overline{z}} \coloneqq \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

Proof. Given $u_x = v_y$ and $u_y = -v_x$,

$$\frac{\partial f}{\partial\overline{z}} = \frac{\partial(u + iv)}{\partial\overline{z}} = \frac{1}{2}(u_x + iu_y + iv_x - v_y) = \frac{1}{2}[(u_x - v_y) + i(u_y + v_x)] = 0.$$

□

The definition of the operator $\partial/\partial\overline{z}$ is motivated by the identities $x = (z + \overline{z})/2$, $y = (z - \overline{z})/2i$, and a formal symbolic application of the chain rule to some $F(x, y)$:

$$\frac{\partial F}{\partial\overline{z}} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial\overline{z}} + \frac{\partial F}{\partial x} \cdot \frac{\partial y}{\partial\overline{z}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right).$$

A complex function f is **analytic**, **regular**, or **holomorphic** in a set S if it is differentiable on an open set containing S (possibly S itself). f is analytic at a point z_0 if it is analytic in a neighborhood of z_0. If f is analytic in the whole plane, then f is **entire**. In particular, polynomials are entire. If f is not analytic at z_0 but *is* in a deleted neighborhood of z_0, then z_0 is a **singular point** or *singularity* of f, e.g. 0 for $1/z$, while $|z|^2$ is nowhere analytic and has no singularities.

Continuity and satisfaction of the Cauchy–Riemann equations are necessary but not sufficient conditions for analyticity in a domain; we have previously seen sufficient condition for differentiability. The sum, product, and composition of analytic functions remain analytic functions.

Proposition 3 (Derivative zero implies constant)**.**

If $f'(z) = 0$ on a domain D, then f is constant on D.

Proof. Note that any two points $P, Q \in D$ are joined by a finite number of line segments in D concatenated.

i. Let $f(z) = u(x, y) + iv(x, y)$. Given $f'(z) = 0$ on D, we have that $u_x + iv_x = 0$, and by the Cauchy–Riemann equations, $v_y - iu_y = 0$. Thus $u_x = u_y = v_x = v_y = 0$ on D.

ii. Next, we check that $u(x, y)$ is constant along any line segment $\overline{PQ}$ in D. Let $s \in [0, 1]$ parametrize $\overline{PQ}$, and let $\vec{u}$ be the unit vector in the direction of $\overline{PQ}$. The directional derivative is then

$$\frac{du}{ds} = (\mathrm{grad}\, u) \cdot \vec{u} = (u_x \hat{\mathbf{i}} + u_y \hat{\mathbf{j}}) \cdot \vec{u} = 0$$

along all of $\overline{PQ}$, which shows that u is constant on the line segment.

iii. Finally, $u(x, y)$ is constant along any path joined by finite number of line segments in D, which proves that u is equal at any two points in D, and thus constant on all of D. By the same argument, $v(x, y)$ is constant on D, so $f = u + iv$ is constant on D.

□

Corollary 2.

If $f(z) = u(x, y) + iv(x, y)$ and $\overline{f(z)}$ are both analytic in a domain D, then $f(z)$ is constant on D.

Proof. We write $U = u$ and $V = -v$, such that $\overline{f} = U + iV$. By analyticity, the Cauchy–Riemann equations hold:

$$u_x = v_y, \quad u_y = -v_x, \quad U_x = V_y, \quad U_y = -V_x.$$

The second set of equations is equivalent to $u_x = -v_y$, $u_y = v_x$. Thus $u_x = 0$ and $v_x = 0$, i.e. $f'(z) = u_x + iv_x = 0$ on all of D. We are done by Proposition 3. □

Definition 1 (Harmonic function)**.**

$H(x, y) \colon \mathbb{R}^2 \to \mathbb{R}$ is **harmonic** on a domain D if it has continuous first- and second-order partial derivatives on D and it satisfies *Laplace's equation* $H_{x,x}(x, y) + H_{y,y}(x, y) = 0$.

Proposition 4 (An analytic function has harmonic components)**.**

If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain D, then u and v are harmonic in D.

Proof. We invoke a result from Chapter 4: if f is analytic at a point, then its components u and v have continuous partial derivatives of all orders at that point. Given that f is analytic in D, the Cauchy–Riemann equations hold:

$$u_x = v_y, \quad u_y = -v_x.$$

Differentiating these equations with respect to both x and y,

$$u_{x,x} = v_{y,x}, \quad u_{y,x} = -v_{x,x}, \quad u_{x,y} = v_{y,y}, \quad u_{y,y} = -v_{x,y}.$$

The continuity of the partial derivatives implies the equality of the mixed partial derivatives: $u_{x,y} = u_{y,x}$, $v_{x,y} = v_{y,x}$. Then $u_{x,x} + u_{y,y} = 0$ and $v_{x,x} + v_{y,y} = 0$, so u and v are harmonic. ☐

Proposition 5 (Analytic iff components are harmonic conjugates)**.**

$f(z) = u(x, y) + iv(x, y)$ is analytic in D iff v is a *harmonic conjugate* of u, that is, if u, v are harmonic in D and their first-order partial derivatives satisfy the Cauchy–Riemann equations on all of D.

Proof. The forward direction follows from Proposition 1 and the previous introduction in *2023-01-13* to the Cauchy–Riemann equations. The converse follows from the result on sufficient results for differentiability. ☐

Proposition 6 (Polar form of Laplace's equation)**.**

Let $f(z) = u(r, \theta) + iv(r, \theta)$ be analytic in a domain not including the origin. Suppose that the partial derivatives of u and v are continuous, and the polar Cauchy–Riemann equations hold. Then

$$r^2 u_{r,r}(r, \theta) + r u_r(r, \theta) + u_{\theta,\theta}(r, \theta) = 0$$

at all points in D, and likewise for $v(r, \theta)$.

Proof. Per Proposition 4, $u_{x,x} + u_{y,y} = 0$ on D. Recall that $u_r = u_x \cos\theta + u_y \sin\theta$. Now, by the chain rule,

$$u_{r,r} = u_{x,r} \cos\theta + u_{y,r} \sin\theta = \left(\frac{\partial}{\partial x} u_x x_r + \frac{\partial}{\partial y} u_x y_r \right) \cos\theta + \left(\frac{\partial}{\partial x} u_y x_r + \frac{\partial}{\partial y} u_y y_r \right) \sin\theta.$$

Recalling that $x_r = \cos\theta$ and $y_r = \sin\theta$, and using the fact that $u_{x,y} = u_{y,x}$ by the continuity of the partial derivatives,

$$\begin{aligned} &= (u_{x,x} \cos\theta + u_{x,y} \sin\theta) \cos\theta + (u_{y,x} \cos\theta + u_{y,y} \sin\theta) \sin\theta \\ &= (\cos^2\theta) u_{x,x} + (2 \cos\theta \sin\theta) u_{x,y} + (\sin^2\theta) u_{y,y}. \end{aligned}$$

Informally, we may write this in operator notation as

$$u_{r,r} = \left(\cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y} \right)^2 u.$$

Similarly, recalling that $u_\theta = -u_x r \sin\theta + u_y r \cos\theta$, let us find

$$\begin{aligned} u_{\theta,\theta} &= -r(u_x \cos\theta + u_{x,\theta} \sin\theta) + r(-u_y \sin\theta + u_{y,\theta} \cos\theta) \\ &= -r u_r - u_{x,\theta} r \sin\theta + u_{y,\theta} r \cos\theta. \end{aligned}$$

Ignoring the $-ru_r$ term in front, by the multivariate chain rule again, this equals

$$\begin{aligned}
-u_{x,\theta} r\sin\theta + u_{y,\theta} r\cos\theta &= -\left(\frac{\partial}{\partial x}u_x x_\theta + \frac{\partial}{\partial y}u_x y_\theta\right) r\sin\theta + \left(\frac{\partial}{\partial x}u_y x_\theta + \frac{\partial}{\partial y}u_y y_\theta\right) r\cos\theta \\
&= -\left(-u_{x,x} r\sin\theta + u_{x,y} r\cos\theta\right) r\sin\theta + \left(-u_{y,x} r\sin\theta + u_{y,y} r\cos\theta\right) r\cos\theta \\
&= r^2\left(u_{x,x}\sin^2\theta - 2u_{x,y}\cos\theta\sin\theta + u_{y,y}\cos^2\theta\right).
\end{aligned}$$

In operator notation, we can express this as

$$u_{\theta,\theta} = \left[-r\frac{\partial}{\partial r} + \left(-\frac{\partial}{\partial x} r\sin\theta + \frac{\partial}{\partial y} r\cos\theta\right)^2\right] u.$$

Now, using the identity $\cos^2\theta + \sin^2\theta = 1$, we observe that

$$u_{r,r} + \frac{1}{r^2}(ru_r + u_{\theta,\theta}) = u_{x,x} + u_{y,y} = 0.$$

□

Corollary 3.

$u_{x,x} + u_{y,y} = u_{r,r} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta,\theta}$.

We finish this chapter by considering how the values of an analytic function on a subdomain or line segment affect or determine its values on the whole domain.

Proposition 7 (Identically zero iff zero in subdomain or line segment)**.**

Suppose that f is analytic on D. If f is zero at each point of a domain or line segment contained in D, then f is identically zero on D.

Proof. Suppose that $f(z_0) = 0$. As D is a connected open set, there exists a zigzag *staircase contour* or *polygonal line* L connecting z_0 to any other point z in D. Let $d > 0$ be the shortest distance from this polygonal line to the boundary of D; if D is the entire plane, take any $d > 0$. Then pick $z_0, z_1, \dots, z_{n-1}, z_n = z$ along L, such that $|z_i - z_{i-1}| < d$ for all i. Define the neighborhoods $N_0, \dots, N_n$ by $N_i :=$ Ball(z_i, d), which are all contained in D.

Borrowing a result from Chapter 6, since f is analytic in the domain N_0, and equals 0 on a subdomain or line segment containing z_0, then f is identically zero on N_0. But $z_1 \in N_0$; inductively applying this argument, we find that $f(z_n) = 0$. But $z_n = z$ was an arbitrary point in D, so f is identically zero on D. □

By considering the (analytic) difference of two functions f, g which are analytic on the same domain D, and coincide in some subdomain or along a line segment contained in D, we find the following natural generalization of Proposition 7.

Proposition 8 (Unique analytic extension from subdomain or line segment)**.**

A function analytic in D is uniquely determined on D by its values in a subdomain or along a line segment contained in D.

Proposition 8 is useful in the problem of extending the domain of an analytic function. If f_1 is analytic in D_1, then there may exist an f_2 analytic in D_2, such that f_1 and f_2 agree on $D_1 \cap D_2$. In this case, f_2 is an **analytic continuation** of f_1 into D_2, and f_2 is unique if it exists by Proposition 8! (We assume D_1 and D_2 are not disjoint, so that $f_1 = f_2$ on $D_1 \cap D_2$ is not a vacuous statement.) Then,

$$F(z) = \begin{cases} f_1(z) & \text{if } z \in D_1 \\ f_2(z) & \text{if } z \in D_2 \end{cases}$$

is a well-defined analytic function on $D_1 \cup D_2$, and f_1, f_2 are called *elements* of F.

Proposition 9 (Reflection principle).

Suppose that f is an analytic in a domain D that contains a segment of the x-axis and is symmetric under reflection over the x-axis. Then $\overline{f(z)} = f(\overline{z})$ for all $z \in D$ iff $f(x) \in \mathbb{R}$ for all x on the segment.

Proof. Suppose that f is real on the segment of the x-axis in D. Let us show that

$$F(z) = \overline{f(\overline{z})} \coloneqq U(x,y) + iV(x,y)$$

is analytic. We see that $U(x,y) = u(x,-y)$ and $V(x,y) = -v(x,-y)$, where $f(z) = u(x,y) + iv(x,y)$. Write $t \coloneqq -y$, and observe that $f(x+it)$ is an analytic function of $x+it$. Invoking the result borrowed from Chapter 4 in the proof of Proposition 4, the first-order partial derivatives are continuous on D and satisfy the Cauchy–Riemann equations

$$u_x = v_t, \quad u_t = -v_x.$$

Reverting the change of variables $y = -t$, we see that $U_x = u_x$, $U_y = -u_t$, $V_x = -v_x$, $V_y = v_t$, i.e.

$$U_x = V_y, \quad U_y = -V_x.$$

Thus F is analytic on D: the first-order partial derivatives of U and V are continuous and satisfy the Cauchy–Riemann equations on D. We also note that by our initial assumption,

$$F(z) = \overline{f(\overline{z})} = f(\overline{z}) = f(z)$$

for every z on the x-axis. By Proposition 8, we actually have $F(z) = f(z)$ on all of D. That is, $\overline{f(\overline{z})} = f(z)$ on D, which is equivalent to $\overline{f(z)} = f(\overline{z})$ above by symmetry. The converse is much simpler: if $\overline{f(z)} = f(\overline{z})$ holds, then for all $z = \overline{z}$ on the x-axis, we see that $iv(x,0) = -iv(x,-0)$, or

$$\overline{f(z)} = \overline{f(\overline{z})} = f(z) = f(\overline{z}).$$

□

For example, $z+1$ and z^2 have the reflection property, as they are real when z is real, while $z+i$ and iz^2 do not have the reflection property on the entire plane.

We will continue with elementary functions in Chapter 3.

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