A bit about the simple random walk

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2023-02-07

Simple random walks need no further introduction. $(S_n)_{n\in\mathbb{N}}$ for $S_n = \sum_{i=1}^n X_i$ the partial sums of i.i.d. Rademachers $X_1, X_2, \ldots$, simple random walks form a canonical example of martingales, Markov chains, and random processes in general. Because its definition is so discrete and so ripe with symmetry, results for simple random walks are often very combinatorial (and thus sometimes asymptotic) in nature — the De Moivre–Laplace Central Limit Theorem, local CLTs, the Law of the Iterated Logarithm, the arcsine law, etc.

To find hitting times and hitting probabilities for simple random walks, a common technique is an alternate form of truncation: halt the random walk at some finite level, which forms a new martingale, the *stopped random walk*. Then, standard arguments are used to pass to the limit: convergence theorems, the monotone continuity of probability, countable unions of events over naturals or rationals, etc.

For example, consider the hitting probabilities in the **symmetric gambler's ruin** problem, which is also a canonical example of an absorbing Markov chain. For levels $-a < 0 < b$, let $\tau_{-a} = \inf\{n \geq 1 : S_n = -a\}$ and τ_b be hitting times, and let $\tau = \tau_{-a} \wedge \tau_b$ be the stopping time, the first time step in which S_n hits $\{-a, b\}$. Invoking the Optional Stopping Theorem, we have that

$$0 = \mathbb{E}(S_0) = \mathbb{E}(S_\tau) = -a\,\mathbb{P}(\tau_{-a} < \tau_b) + b\,\mathbb{P}(\tau_b < \tau_{-a}) = -a \cdot (1-\alpha) + b \cdot \alpha,$$

which shows that the hitting probability of level b is $\alpha = \frac{a}{a+b}$. If $a = -1$ and $b = M - 1$, this equality says that

$$\mathbb{P}_1(\tau_M < \tau_0) = \frac{1}{M}.$$

The probability that starting from 1, the trajectory of the process ever reaches M before it hits 0 is exactly inversely proportional to M, the distance between 0 and M. For this particular case, this is a stronger statement than Doob's martingale inequality, the Markov-like bound on the tail probability of the maximal process.

We can also consider the **expected hitting time** $\mathbb{E}(\tau)$ in the symmetric gambler's ruin problem. Again, we will need to consider the stopped process $S_{n\wedge\tau}$, because the regular S_n does not incorporate any information about τ. We know that $S_{n\wedge\tau}^2 - (n \wedge \tau)$ is a martingale, which implies that

$$\mathbb{E}(S_{n\wedge\tau}^2 - (n\wedge\tau)) = \mathbb{E}(S_0^2 - 0) = 0.$$

Passing to the limit as $n \to \infty$, $\mathbb{E}(n \wedge \tau) \to \mathbb{E}(\tau)$ by MCT, and $\mathbb{E}(S_{n\wedge\tau}^2) \to \mathbb{E}(S_\tau^2)$ by BCT, where $S_\tau \leq \max(a, b)$. Therefore the expected hitting time is

$$\mathbb{E}(\tau) = \mathbb{E}(S_\tau^2) = \frac{b}{a+b} \cdot a^2 + \frac{a}{a+b} \cdot b^2 = ab.$$

Let us use the above to show that starting from 0, the random walk S_n will hit any given level $b > 0$ almost surely. That is, we wish to pass from a two-sided bound to a one-sided bound. We have previously seen a proof of this fact in the context of martingales with bounded increments, where we showed that

$$\mathbb{P}\left(M_n \text{ has a finite limit or } \liminf_{n\to\infty} M_n = -\infty \wedge \limsup_{n\to\infty} M_n \right) = 1.$$

Of course, by its nature, the random walk S_n cannot have a limit, so S_n must in fact oscillate infinitely often, visiting every level [!] infinitely many times almost surely.

But, to use the fact that $\mathbb{E}(\tau_{-a} \wedge \tau_b) = ab$, we observe that

$$\mathbb{P}(\tau_b < \infty) \geq \mathbb{P}(\tau_b < \tau_{-a}) \quad \text{for any } -a < 0.$$

$\mathbb{E}(\tau) < \infty$ implies that $\tau = \tau_{-a} \wedge \tau_b < \infty$ a.s., so if $\tau_b < \tau_{-a}$, then $\tau_b = \tau < \infty$. After reducing to the simpler case of the "strip" $[-a, b]$ bounded above and below, we can use the previously found hitting probability:

$$\mathbb{P}(\tau_b < \infty) \geq \lim_{a\to\infty} \mathbb{P}(\tau_b < \tau_{-a}) = \lim_{a\to\infty} \frac{a}{a+b} = 1.$$

This shows that the symmetric random walk is *recurrent*. Combined with the fact that $\mathbb{E}(\tau_b) = \infty$ for any b, again through a limiting argument, we have shown that the symmetric random walk is *null recurrent*. With some more analysis, we can gain a finer understanding of τ_0 than $\mathbb{E}_1(\tau_0) = \infty$:

$$\mathbb{P}_0(\tau_0 > 2n) = \mathbb{P}_0(S_{2n} = 0) \sim \frac{1}{\sqrt{\pi n}}.$$

This note was adapted largely from Professor Shirshendu Ganguly's spring 2023 iteration of Math C218B / Stat C205B at UC Berkeley.

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