Two 0–1 laws

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Beyond various limit theorems, convergence theorems, and concentration inequalities, another class of useful results in probability are **0–1 laws**, which provide sufficient conditions for an event to be *trivial*, $\mathbb{P}(\cdot) = 0$ or 1. In applications, *any* lower bound or upper bound is enough to show that such an event is almost sure or almost never.

One of the most common arguments for showing that a σ-algebra $\mathcal{F}$ is trivial is to show that $\mathcal{F}$ is independent of itself: $\mathbb{P}(A) = \mathbb{P}(A \cap A) = \mathbb{P}(A) \cdot \mathbb{P}(A) = \mathbb{P}(A)^2$ iff $\mathbb{P}(A) \in \{0, 1\}$.

Theorem 1 (Kolmogorov's 0–1 law)**.**

Let $X_1, X_2, \ldots$ be a sequence of independent random variables, and let

$$\mathcal{T} = \bigcap_{n=1}^{\infty} \sigma(X_n, X_{n+1}, \ldots)$$

be the tail σ-algebra, analogous to the limit superior of the information in $(X_n)_{n \geq 1}$. Then $\mathcal{T}$ is trivial: every $A \in \mathcal{T}$ has $\mathbb{P}(A) = 0$ or 1.

Proof. It suffices to show that $\mathcal{T}$ is independent of $\sigma(X_1, X_2, \ldots) \supseteq \mathcal{T}$. And, as $\bigcup_{n=1}^{\infty} \sigma(X_1, \ldots, X_n)$ is a π-system generating $\sigma(X_1, X_2, \ldots)$, it suffices to show that $\mathcal{T} \perp\!\!\!\perp \bigcup_{n=1}^{\infty} \sigma(X_1, \ldots, X_n)$. For any $B \in \sigma(X_1, \ldots, X_n)$,

$$A \in \mathcal{T} \subseteq \sigma(X_{n+1}, X_{n+2}, \ldots) \implies A \perp\!\!\!\perp B.$$

$\square$

The following 0–1 law is a stronger result showing the triviality of a larger σ-algebra.

Theorem 2 (Hewitt–Savage 0–1 law)**.**

Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, and let

$$\mathcal{E} = \{A \in \sigma(X_1, X_2, \ldots) : A \text{ invariant under any finite permutation of the indices}\}$$

be the exchangeable σ-algebra. Then $\mathcal{E}$ is trivial.

Proof. To show that $\mathcal{E}$ is independent of itself, it suffices to show that $\mathbb{E}(f(X_1,\ldots,X_k) \mid \mathcal{E}) = \mathbb{E}(f(X_1,\ldots,X_k))$ for every $k \geq 1$ and bounded measurable function f, in which case $\mathcal{E} \perp\!\!\!\perp \bigcup_{k=1}^{\infty} \sigma(X_1,\ldots,X_k)$. Let

$$\mathcal{E}_n = \{A \in \sigma(X_1, X_2, \ldots) : A \text{ invariant under every permutation } \pi \in S_n\},$$

such that $\mathcal{E} = \bigcap_{n=1}^{\infty} \mathcal{E}_n$.

i. We observe that $\mathbb{E}(f(X_1) \mid \mathcal{E}_n) = \mathbb{E}(f(X_i) \mid \mathcal{E}_n)$ for all $i = 1, \ldots, n$ by a simple change of measure, along with the i.i.d.ness of the X_i and the invariance of events in $\mathcal{E}_n$. Then

$$\mathbb{E}(f(X_1) \mid \mathcal{E}_n) = \frac{1}{n}\sum_{i=1}^{n} f(X_i),$$

where the right-hand side is $\mathcal{E}_n$-measurable and equal to the left-hand side by linearity of conditional expectation. Using the same argument,

$$\mathbb{E}(f(X_1,\ldots,X_k) \mid \mathcal{E}_n) = \frac{1}{\binom{n}{k}} \sum_{1\leq i_1 < \cdots < i_k \leq n} f(X_{i_1},\ldots,X_{i_k}).$$

ii. Let $M_n \coloneqq \mathbb{E}(f(X_1,\ldots,X_k) \mid \mathcal{E}_n)$. We observe that $(M_n)_{n\to\infty}$ is a bounded *backwards martingale*, which makes it automatically uniformly integrable, with limit

$$\mathbb{E}(f(X_1,\ldots,X_k) \mid \mathcal{E}_n) \to M \coloneqq \mathbb{E}(f(X_1,\ldots,X_k) \mid \mathcal{E}) \quad \text{a.s. and in } L^1.$$

iii. But, by step i, the right-hand side M is also equal to the $\mathcal{T}$-measurable

$$\lim_{n\to\infty} \frac{1}{\binom{n}{k}} \sum_{1\leq i_1 < \cdots < i_k \leq n} f(X_{i_1},\ldots,X_{i_k}).$$

Tail-measurable random variables are trivial, so M is in fact constant almost surely. Because

$$\mathbb{E}(M_n) = \mathbb{E}(\mathbb{E}(f(X_1,\ldots,X_k) \mid \mathcal{E}_n)) \equiv \mathbb{E}(f(X_1,\ldots,X_k))$$

for every n by the law of iterated expectation, $\mathbb{E}(M_n) \to \mathbb{E}(M)$ must equal $\mathbb{E}(f(X_1,\ldots,X_k))$, by which we are done: we have shown that $M = \mathbb{E}(f(X_1,\ldots,X_k) \mid \mathcal{E})$ equals $\mathbb{E}(f(X_1,\ldots,X_k))$ almost surely.

□

We have already seen the Borel–Cantelli lemmas, but Blumenthal's and Lévy's 0–1 laws are yet to be covered. We will leave them to another day.

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