Spectral Topological Spaces as Spectra of Rings

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1 Introduction

Assume throughout that rings are commutative and unital. Recall that Spec defines a contravariant functor from Ring to Top. We are interested in a simple characterization of the essential image of Spec, which would allow us to fully characterize the topologies underlying ring spectra, and more generally schemes. Towards this end, we make the following definition.

Notation 1.1. Let $Q(X)$ denote the collection of quasicompact open subsets of X.

Definition 1.2. A topological space X is **spectral** if the following conditions are satisfied.

i. The space X is quasicompact.

ii. The space X is quasiseparated: $Q(X)$ is closed under finite intersection.

iii. The collection $Q(X)$ forms a basis for the topology of X.

iv. The space X is sober: Every irreducible component of X has a unique generic point.

The main goal of this paper is to explain the proof of the following theorem.

Theorem 1.3 (Hochster [3])**.** *Let X be a topological space. The following are equivalent:*

1. *The space X is spectral.*
2. *The space X is homeomorphic to* Spec A *for some ring A.*

The implication (2) $\implies$ (1) is well known, so the crux of the theorem lies in the converse implication (1) $\implies$ (2). In §2, we give a proof of (1) $\implies$ (2) by explicitly constructing a ring whose spectrum is a given spectral space. This construction will turn out to be functorial. In §3, we explore the consequences of functoriality by inverting Spec on certain subcategories of spaces; provide an alternate characterization of spectral spaces as the projective limits of finite sober spaces; and extend the main result to schemes in general. The results and most of the proofs in this paper are taken from [3].

2 Main result

2.1 Topological preliminaries

Let us first establish some definitions pertaining to the topological structure of a spectral space.

Definition 2.1. A map $f\colon X \to Y$ of spectral spaces is **spectral** if for every $V \in Q(Y)$, we have that $f^{-1}(V) \in Q(X)$.

Notation 2.2. Let $\mathsf{STop} \subset \mathsf{Top}$ denote the category of spectral spaces and spectral maps.

Henceforth, we will regard Spec as a functor from $\mathsf{Ring}^{\mathrm{op}}$ to STop.

Definition 2.3. We say that Spec is **invertible** on a subcategory S of STop if there exists a functor $G\colon \mathsf{S} \to \mathsf{Ring}^{\mathrm{op}}$ such that $\mathrm{Spec} \circ G$ is isomorphic to the inclusion functor $\mathsf{S} \hookrightarrow \mathsf{STop}$. We will call G a **space-preserving** functor.

Now, fix a spectral space X.

Definition 2.4. The **patch topology** (or constructible topology) on X is the topology with open subbasis $\{U, U^c \mid U \in Q(X)\}$. We will call a set closed in the patch topology a **patch**.

Lemma 2.5. *The patch topology on X is compact (= quasicompact + Hausdorff).*

Proof. [4, Lemma 5.23.2] Since X is sober, any two distinct points $x, y \in X$ can be separated by some open set U, which is clopen in the patch topology. This shows Hausdorffness. To show quasicompactness, we can reduce to considering the subbasis $\{U, U^c \mid U \in Q(X)\}$ by the Alexander subbase theorem. We will show that every family $\{E_i\}_{i\in I}$ in the subbasis having the finite intersection property intersects.

A standard application of Zorn's lemma allows us to assume that $\{E_i\}_{i\in I}$ is a maximal family. Then, a standard topological argument by contradiction shows that the intersection Z of the patches among $\{E_i\}_{i\in I}$ is irreducible (and a patch). The generic point of Z then belongs to $\bigcap_{i\in I} E_i$, which shows quasicompactness. □

Lemma 2.6. *If Y is a patch in X, then $x \in \mathrm{cl}\, Y$ if and only if x belongs to the closure of some point in Y.*

Proof. Suppose that $x \in \mathrm{cl}\, Y$. Let $\{U_i\}_{i\in I}$ be the collection of quasicompact open neighborhoods of x. From our assumptions, for each $i \in I$, the intersection $U_i \cap Y$ is a nonempty patch. Because the patch topology is quasicompact, we find that the intersection $\bigcap_{i\in I}(U_i \cap Y)$ is nonempty. We conclude that x belongs to the closure of any point in $\bigcap_{i\in I} U_i \cap Y$. Conversely, we know that the closure of any point $y \in Y$ is contained in $\mathrm{cl}\, Y$. □

Motivated by the preceding lemma, we also introduce the following notation.

Notation 2.7. Let $\Sigma(X)$ denote the specialization order on X, i.e., the set of all pairs $(y, x) \in X \times X$ such that $x \in \mathrm{cl}\{y\}$.

2.2 Springs

As an intermediary between spaces and rings, let us define springs (= 'space' + 'ring') as follows.

Definition 2.8. A **spring** is a pair (X, A) consisting of a reduced ring A and a dense patch X of Spec A. A morphism of springs $(X, A) \to (X', A')$ is a ring homomorphism $\varphi\colon A' \to A$ such that $(\operatorname{Spec}\varphi)(X) \subseteq X'$.

Notation 2.9. Let Spr denote the category of springs and spring morphisms.

We will also find it useful to introduce "additional" structure into a spring (X, A) in the form of quotients of A by prime ideals.

Lemma 2.10. *We identify Spr with an equivalent category (of the same name), which consists of the following data. The objects are triples of the form $(X, \{A^x\}_{x\in X}, A)$, where X is a spectral space, $\{A^x\}_{x\in X}$ is a family of integral domains indexed by X, and A is a subring of $\prod_{x\in X} A^x$, such that*

1. *For every $x \in X$, $A^x = \{a(x) \mid a \in A\}$.*
2. *For every $a \in A$, the set $d(a) := \{x \in X \mid a(x) = 0\}$ belongs to $Q(X)$.*
3. *The collection $\{d(a)\}_{a\in A}$ forms a basis for the topology of X.*

The morphisms are pairs (f, φ) consisting of a map $f\colon X \to X'$ and a ring homomorphism $\varphi\colon A' \to A$, such that for every $a' \in A'$, we have that $f^{-1}(d(a')) = d(\varphi(a'))$.

Proof. Recall that a functor is an equivalence of categories if and only if it is fully faithful and essentially surjective. Let us consider the obvious functor $F\colon (X, A) \mapsto (X, \{A^x\}_{x\in X}, A)$. The functor F is well-defined: it is straightforward to verify the three properties listed above for $F(X, A)$, where A^x is taken to be the quotient A/x for the prime ideal $x \in \operatorname{Spec} A$. The full-faithfulness of F also follows from the definition of spring morphisms and known properties of Spec. To show that F is essentially surjective, let $(X, \{A^x\}_{x\in X}, A)$ be a triple specified above. We identify each point $x \in X$ with the prime ideal $\iota(x) := \{a \in A \mid a(x) = 0\} \in \operatorname{Spec} A$. It follows that $(\iota(X), A)$ is a spring, from which the given triple $(X, \{A^x\}_{x\in X}, A)$ arises as $F(\iota(X), A)$. □

Notation 2.11. For convenience, we write $\mathbf{A}$ for the spring (X, A) (equivalently $(X, \{A^x\}_{x\in X}, A)$), and $\mathbf{A}'$ for (X', A').

Notation 2.12. We also write $z(a) := X \setminus d(a) = \{x \in X \mid a(x) \neq 0\}$. Recall that $D(a) = \{\mathfrak{x} \in \operatorname{Spec} A \mid a \notin \mathfrak{x}\}$ and $V(a) = \operatorname{Spec}(A) \setminus D(a)$.

Definition 2.13. We call a spring (X, A) **affine** if $X = \operatorname{Spec} A$.

Proposition 2.14. *The following are equivalent:*

1. *The spring (X, A) is affine.*
2. $(*)$ *For every finite subset B of A and every element $a \in A$, if $\bigcap_{b\in B} z(b) \subseteq z(a)$, then $a \in \operatorname{rad} B$.*

Proof. The implication (1) $\Longrightarrow$ (2) follows from a standard property of ring spectra. Conversely, assume $(*)$, but suppose instead that $X \neq \operatorname{Spec} A$. Since X is a patch in Spec A, there exists a finite subset B of A and an element $a \in A$ such that $X \subseteq U_{a,B} := V(a) \cup (\bigcup_{b\in B} D(b))$, where $U_{a,B}$ is also a proper subset of Spec A. However, from $(*)$, it follows that $a \in \operatorname{rad} B$. This implies that $U_{a,B} = \operatorname{Spec} A$, a contradiction. □

Our eventual goal is to produce an affine spring $\mathbf{A}$ from the spectral space X, but this is not a straightforward task. Instead, we will first obtain a so-called *simple* spring $\mathbf{A}$, whose ring we will then *extend* until condition $(*)$ is satisfied, as we explain in the next section.

2.3 Indexed extensions

Definition 2.15. An **index** on a spring (X, A) is a collection of additive valuations $v_{y,x}\colon \operatorname{Frac}(A^y) \to \mathbb{Z} \cup \{\infty\}$ indexed by $\Sigma(X)$ satisfying the following properties.

1. For every element $a \in A$ and pair $(y, x) \in \Sigma(X)$ such that $y \in d(a)$, we have that $v_{y,x}(a) \geq 0$ with equality if and only if $a(x) \neq 0$. (We write $v_{y,x}(a) := v_{y,x}(a(y))$.)
2. For every element $a \in A$, there exists a positive integer N such that for every pair $(y, x) \in \Sigma(X)$ with $y \in d(a)$, we have that $v_{y,x}$ is bounded above on A^y by N. (We will write this as $v_{y,x}(A^y) \leq N$.)

We denote this indexed spring by $(\mathbf{A}, v)$ (or simply $\mathbf{A}$).

Definition 2.16. A morphism of indexed springs is a spring morphism (f, φ) such that for each element $a' \in A'$ and pair $(y, x) \in \Sigma(X)$ with $y \in d(\varphi(a'))$, we have that $v_{y,x}(\varphi(a')) = v_{f(y),f(x)}(a')$.

Notation 2.17. Let IndSpr denote the category of indexed springs.

Definition 2.18. Suppose that $\mathbf{A}$ and $\mathbf{A}'$ are springs such that $X = X'$. We say that $\mathbf{A}'$ is an **extension** of $\mathbf{A}$ if for every $x \in X$, the integral domain A'^x is a subring of $\operatorname{Frac}(A^x)$ that contains A^x. If $(\mathbf{A}, v)$ is an indexed spring such that v remains an index for $\mathbf{A}'$, then $\mathbf{A}'$ is called a v**-extension** of $\mathbf{A}$. We say that an element $b \in \prod_{x\in X} \operatorname{Frac}(A^x)$ **induces** a (v-)extension of $\mathbf{A}$ if $(X, A[b])$ is a spring extending $\mathbf{A}$.

Notation 2.19. Let $\mathbf{A}$ be a spring. Suppose that $a, b \in A$ are elements with $z(b) \subseteq z(a)$. Let $a \oslash b$ denote the element of $\prod_{x\in X} \operatorname{Frac}(A^x)$ which takes the value $a(x)/b(x)$ on $x \in z(b)$ and 0 on $x \in d(b)$.

The purpose of introducing the family of bounded valuations we call an index is to allow us to reasonably extend springs which are not necessarily affine by adjoining "quotients" of the form $a \oslash b$. In particular, we find the following key equivalent condition, which is framed in terms of the values of the valuations $\{v_{y,x}\}_{(y,x)\in\Sigma(X)}$.

Proposition 2.20. *Let $(\mathbf{A}, v)$ be an indexed spring, and let a, b be elements of A. Supposing that $z(b) \subseteq z(a)$, the following are equivalent:*

1. *The element $a \oslash b$ induces a v-extension of $(\mathbf{A}, v)$.*
2. *(†) For every pair $(y, x) \in \Sigma(X)$ such that $y \in d(a)$, we have that $v_{y,x}(A^y) \geq v_{y,x}(b)$, with equality if and only if $x \in d(b)$.*

Proof. If (1) is the case, then for any pair $(y, x) \in \Sigma(x)$ with $y \in d(a) \subseteq d(b)$, we know that $v_{y,x}(b(y)) \geq 0$ with equality if and only if $b(x) = 0$ by the definition of an index. Since $v_{y,x}$ is nonnegative on A^y, condition (2) follows. Now, assume (2). It suffices to prove that for every $p = \sum_{i=0}^{m} a_i (a \oslash b)^i$, we have that $d(p)$ is quasicompact and open. Let $q := b^m p$. We observe that p vanishes on the set

$$d(p) = (d(a) \cap d(q)) \cup (z(a) \cap d(a_0)).$$

Thus, $d(p)$ is a patch. It remains to show that $d(p)$ is open, or equivalently that

$$\begin{aligned} z(p) &= (d(a) \cap z(q)) \cup (z(a) \cap z(a_0)) \\ &= (d(b) \cap z(q)) \cup (z(b) \cap z(a_0)) \end{aligned}$$

is closed. Let $y \in z(p)$ and $x \in \operatorname{cl}\{y\}$ (i.e., $(y, x) \in \Sigma(X)$). We may assume without loss of generality that $y \in d(a) \cap z(q)$ and $x \in z(b) \subseteq z(a)$, since we know that sets of the form $z(\cdot)$ are closed. By (2), we have that $v_{y,x}(a) > v_{y,x}(b)$. Then $p(y)$ evaluates to 0, and we find by the standard properties of valuations and (2) that

$$\begin{aligned} v_{y,x}(-a_0(y)b(y)^m) &= m v_{y,x}(b) + v_{y,x}(a_0) \\ v_{y,x}(p(y) - a_0(y)b(y)^m) &> m v_{y,x}(b). \end{aligned}$$

That is, $v_{y,x}(a_0) > 0$. By the definition of an index, it follows that $x \in z(a_0)$. Since $x \in z(a_0) \cap z(b) \subseteq z(p)$, we conclude that $z(p)$ is closed. Lastly, we observe that the two properties of an index also hold for $(A[a \oslash b], v)$. □

Notation 2.21. Let $E(\mathbf{A}, v)$ denote the set of elements $a \oslash b \in \prod_{x \in X} \mathrm{Frac}(A^x)$ that induce a v-extension of A.

Corollary 2.22. *The set $E(\mathbf{A}, v)$ induces a v-extension $\mathbf{A}^{(1)}$ of $\mathbf{A}$.*

Proof. The key observation is that condition (†) does not depend on whether we consider a and b as elements of A or a v-extension of A. Thus, adjoining the elements of $E(\mathbf{A}, v)$ produces a (v-)extension of $\mathbf{A}$. □

Notation 2.23. For every $n \geq 1$, let $\mathbf{A}^{(n+1)}$ be the v-extension of $\mathbf{A}^{(n)}$ induced by $E(\mathbf{A}^{(n)}, v)$.

Construction 2.24. Let $C(\mathbf{A}) := C(\mathbf{A}, v)$ be the closure of $\mathbf{A}$ under the operation of adjoining an element of the form $a \oslash b$ whenever it induces a v-extension. Equivalently, $C(\mathbf{A})$ is the v-extension of $\mathbf{A}$ whose underlying space is still X, and whose ring is $\bigcup_{n \geq 1} A^{(n)}$.

In fact, we can also define C on morphisms of indexed springs in a natural way, as we explain. Let $(f, \varphi)\colon (\mathbf{A}, v) \to (\mathbf{A}', v')$ be a morphism of indexed springs. Let $a' \oslash b' \in E(\mathbf{A}', v')$. Then we claim that there exists a unique extension $(f, \varphi^{(1)})$ of (f, φ) as an indexed spring morphism from $(\mathbf{A}[\varphi(a') \oslash \varphi(b')], v)$ to $(\mathbf{A}'[a' \oslash b'], v')$.

Moreover, let B' be any v'-extension of $\mathbf{A}[\varphi(a') \oslash \varphi(b')]$, let B be any v-extension of A, and let $(f, \psi)\colon B \to B'$ be an indexed spring morphism such that $\psi_{\restriction A'} = \varphi$. Then $\varphi(a') \oslash \varphi(b') = \psi(a') \oslash \psi(b')$ in fact equals $\psi(a' \oslash b') \in B$. Thus, adjoining $E(\mathbf{A}', v')$ to $\mathbf{A}'$, we obtain an extension of (f, φ) as a morphism from $(\mathbf{A}^{(1)}, v)$ to $(\mathbf{A}'^{(1)}, v')$. Inductively extending to a morphism $\mathbf{A}^{(n)} \to \mathbf{A}'^{(n)}$, it follows that there is a natural extension of (f, φ) to some $(f, \varphi^C)\colon (C(\mathbf{A}), v) \to (C(\mathbf{A}'), v')$. We thus define $C(f, \varphi)$ to be (f, φ^C). □

We had defined a space-preserving functor in Definition 2.3, but we will also call a functor space-preserving if it preserves spaces in the obvious sense, i.e., if it acts as the identity functor for the space associated to any object (and any map of spaces) in its domain.

Lemma 2.25. *The map of categories $C\colon$ IndSpr $\to$ IndSpr is a space-preserving functor.*

Proof. We have that C is space-preserving by construction. To show that C is a functor, it suffices to verify the claims made in Construction 2.24. We will proceed similarly to the proof of Proposition 2.20. Given $(f, \varphi)\colon (\mathbf{A}, v) \to (\mathbf{A}', v')$ and $a' \oslash b' \in E(\mathbf{A}, v)$ as above, we naturally define the ring homomorphism $\varphi^{(1)}\colon \mathbf{A}'[a' \oslash b'] \to \mathbf{A}[\varphi(a') \oslash \varphi(b')]$ by setting $\varphi^{(1)}$ to be φ on A' and mapping generator to generator. Since $f^{-1}(d(a)) = d(\varphi(a))$ for every $a \in A$, we see that $(f, \varphi^{(1)})$ is indeed correctly defined, where explicitly $\varphi^{(1)}(a' \oslash b') = \varphi(a') \oslash \varphi(b')$. A nearly identical check shows that $\psi(a' \oslash b')$ equals the element $\psi(a') \oslash \psi(b')$. □

Now, we find that the key property held by the "closure functor" $C\colon$ IndSpr $\to$ IndSpr is that it extends any arbitrary indexed spring to an indexed spring which is "nearly" affine. More precisely, we have the following proposition.

Proposition 2.26. *Let $(\mathbf{A}, v)$ be an indexed spring. Then $C(\mathbf{A})$ satisfies the following property:*

$$(**) \qquad \text{For all elements } a, b \in C(A), \text{ if } z(b) \subseteq z(a), \text{ then } a \in \operatorname{rad} b.$$

Proof. Let $a, b \in C(A) = \bigcup_{n \geq 1} A^{(n)}$, and let n be sufficiently large so that $a, b \in A^{(n)}$. Suppose that $z(b) \subseteq z(a)$. By Definition 2.15, we can choose a positive integer N such that for each pair $(y, x) \in \Sigma(X)$ with $y \in d(b)$, we have that $v_{y,x}(b) \leq N$. Then we check that $a^{N+1} \oslash b$ induces a v-extension of $\mathbf{A}$ by checking condition (†) as follows. Suppose $(y, x) \in \Sigma(X)$ is such that $y \in d(a^{N+1})$; since A is a reduced ring by definition, we see that $d(a^{N+1}) = d(a) \subseteq d(b)$, so y belongs to $d(b)$. Then because $v_{y,x} \leq N$, we have that $v_{y,x}(a^{N+1}) = (N+1)v_{y,x}(a) \geq v_{y,x}(b)$ with equality if and only if $x \in d(b)$, as desired. □

Remark 2.27. Compare $(**)$ to the (stronger) affine spring condition $(*)$ in Proposition 2.14. In the next section, we will develop a mild finiteness condition under which $(**)$ implies $(*)$, and we will easily satisfy this finiteness condition by construction, thus producing an affine spring as desired.

2.4 Proof of Hochster's theorem

In this section, we will associate functorially to a given spectral space X a polynomial ring A and an index v. We will then show that the closure $C(A)$ of this indexed spring under indexed extensions is affine. This will allow us to conclude Theorem 1.3 by composing functors, and provide a more generally applicable method of inverting Spec.

Notation 2.28. Henceforth, fix an arbitrary field $\Bbbk$. (Our construction will be independent of the choice of $\Bbbk$.)

Definition 2.29. A **space with indeterminates** is a triple (X, I, s) consisting of a spectral space X, an index set I (together with algebraically independent indeterminates $\{t_i\}_{i\in I}$), and a choice of open subbasis $s\colon I \to Q(X)$. A morphism of such spaces is a pair (f, g) consisting of a spectral map $f\colon X \to X'$ and an *injective* set function $g\colon I \to I'$ such that $s' \circ g = f^{\text{pre}} \circ s$.

Notation 2.30. Let STopIt denote the category of spaces with indeterminates.

Notation 2.31. For convenience, we will denote each open set $s(i)$ by O_i.

Remark 2.32. To motivate the following construction, recall that for $X = \operatorname{Spec} A$, the collection of open sets $\{D(a) \in Q(X) \mid a \in A\}$ forms a basis for the topology of X. In other words, $Q(X)$ (loosely) corresponds to the elements of A. Informally, this initial picture of A is subsequently refined by the index, which captures the information present in $\Sigma(X)$, namely the inclusion order of prime ideals.

Construction 2.33. Let us define a space-preserving (in the obvious sense) functor $G\colon \mathsf{STopIt} \to \mathsf{IndSpr}$ as follows. Let (X, I, s) be a given space with indeterminates. Define $R = \Bbbk[t_i]_{i\in I}$. For each $i \in I$, let $T_i\colon X \to R$ be the function which takes the value t_i on O_i and 0 elsewhere. Define A to be the polynomial ring $\Bbbk[T_i]_{i\in I} \subset R^X$. We claim that $(X, \{A^x\}_{x\in X}, A)$ is a spring, where $A^x = \{a(x) \mid a \in A\}$ is equivalently $\Bbbk[t_i]_{i\in I : x\in O_i}$. Now, for each pair $(y, x) \in \Sigma(X)$, we define the valuation $v_{y,x}$ on $\operatorname{Frac}(A^y)$ to be the unique valuation satisfying the following properties:

i. $v_{y,x}(t_i) = 0$ if $x \in O_i$ and 1 otherwise.

ii. Let $\{i_1, \ldots, i_n\}$ be a finite subset of I. If p is any linear combination of monomial products $\{p_j\}_{j=1}^m$ of $t_{i_1}, \ldots, t_{i_n}$ with nonzero coefficients, then $v_{y,x}(p) = \min_{j=1}^m \{v_{y,x}(p_j)\}$.

We define $G(X, I, s)$ to be the indexed spring $(\mathbf{A}, v)$. Now, let us define G on a morphism of spaces of indeterminates $(f, g)\colon (X, I, s) \to (X', I', s')$. There is a unique homomorphism of $\Bbbk$-algebras $\varphi_0\colon \Bbbk[t_i]_{i\in I} \to \Bbbk[t_{i'}]_{i'\in I'}$ given by the assignment $t_i \mapsto t_{g(i)}$, which induces a homomorphism $\varphi\colon \Bbbk[T_i]_{i\in I} \to \Bbbk[T_{i'}]_{i'\in I'}$. We claim that (f, φ) is a morphism of indexed springs, and we define $G(f, g)$ to be (f, φ). □

Moreover, we observe that the constructed indexed spring $G(X, I, s)$ enjoys a finiteness condition, which we will define for springs in general below.

Definition 2.34. Suppose that $\mathbf{A}$ is a spring such that all of the integral domains A^x are subrings of a common ring R, i.e., $A \subset \prod_{x\in X} A^x$ can be treated as a subring of $R^X = \prod_{x\in X} R$. Then $\mathbf{A}$ is **simple** if for every $a \in A$, the set of images $\{a(x) \mid x \in X\} \subseteq R$ is finite.

Lemma 2.35. *$G\colon \mathsf{STopIt} \to \mathsf{IndSpr}$ is a space-preserving functor whose image objects are simple indexed springs.*

Proof. The map of categories G is space-preserving, with simple image objects, by construction. To show that G is a functor, it suffices to verify that $G(X, I, s) = (\mathbf{A}, v)$ is an indexed spring, and that $G(f, g) = (f, \varphi)$ is a morphism of indexed springs. The first claim is straightforward, and the argument is more easily illustrated with an example: for the element $a = T_1 T_2^5 - T_3 + 2T_4$, the set $d(a)$ equals $(O_1 \cup O_2) \cap O_3 \cap O_4$. It is clear then that $\{d(a)\}_{a\in A} \supseteq Q(X)$ forms a basis for the topology of X. To see that v is an index, we note that any valuation of any element $a \in A$ is nonnegative and at most $\deg a$ by construction. This shows that $(\mathbf{A}, v)$ is an indexed spring. The second claim is also straightforward: it suffices to check that $f^{-1}(d(a')) = d(\varphi(a'))$ for a' of the form $T_{i'}$, which follows from the relevant definitions. We note the following facts: $d(\prod_{j\in J} T_{i_j}) = \bigcap_{j\in J} O_{i_j}$, and with notation as in Construction 2.33, $d(p(T_{i_1}, \ldots, T_{i_n})) = \bigcup_{j=1}^m d(p_j(T_{i_1}, \ldots, T_{i_n}))$. □

The importance of G having simple image objects is justified by the following proposition, which will be the last piece of the puzzle that we need.

Proposition 2.36. *If $(\mathbf{A}, v)$ is a simple indexed spring, then $C(\mathbf{A}, v)$ is an affine indexed spring.*

Proof. If $(\mathbf{A}, v)$ is simple, we observe that $C(\mathbf{A}, v)$ is also simple, i.e., every element has finite image. Now, we show that for a simple indexed spring $(\mathbf{A}, v)$, condition $(**)$ implies condition $(*)$, as follows. It suffices to show that for any finite subset B of A, there exists an element $c \in \langle B \rangle$ such that $z(c) = \bigcap_{b \in B} z(b)$, or equivalently $d(c) = \bigcup_{b \in B} d(b)$. By the simplicity of $(\mathbf{A}, v)$, we can divide the set $\bigcup_{b \in B} d(b)$ into finitely many regions $Y_1, \ldots, Y_n$, such that on each region, every $b \in B$ is constant (as a function on X). Choose a point y_i from each region Y_i, and let $Y := \{y_1, \ldots, y_n\}$. Note that the set of images $\{c(x) \mid x \in X\}$ is the same as $\{c(y) \mid y \in Y\}$. Thus, it suffices to find an element $c \in \langle B \rangle$ that does not vanish on Y, i.e., such that none of the prime ideals $y_1, \ldots, y_n$ contain c, which we can find. By Proposition 2.26, $C(\mathbf{A}, v)$ satisfies condition $(**)$, so we conclude that $C(\mathbf{A}, v)$ is affine. □

Putting everything together, we arrive at the following proof of the main result.

Proof of Theorem 1.3. It suffices to exhibit an inverse to Spec on an arbitrary interval diagram $\mathbb{I}$ in STop as follows. First, we embed $\mathbb{I}$ into STopIt by the (space-preserving) functor J to obtain the following diagram.

$$\circlearrowright (X, Q(X), \mathrm{id}_{Q(X)}) \xrightarrow{(f, \mathrm{id}_{Q(X)} \times f^{-1})} (X', Q(X) \times Q(X'), (U, V) \mapsto V) \circlearrowleft$$

Now, let the desired functor $F\colon \mathsf{STopIt} \to \mathsf{Ring}$ be the composition

$$\mathsf{STopIt} \xrightarrow{G} \mathsf{IndSpr} \xrightarrow{C} \mathsf{IndSpr} \xrightarrow{\text{forget}} \mathsf{Ring}.$$

We observe that F is space-preserving. The spaces with indeterminates X and X' are mapped by the functor G to simple indexed springs, which the functor C extends to affine springs. Thus, the forgetful functor from IndSpr to Ring indeed maps to rings A and A' whose spectra are X and X' respectively. □

2.5 Example of construction

Let us explicitly perform Hochster's construction for a simple example.

Example 2.37. Let $X = \{\eta, \mathfrak{m}\}$ be the **Sierpiński space**, where η is the generic point and $\mathfrak{m}$ a closed point. For convenience, we write $U = \{\eta\}$, so that $Q(X) = \{X, U, \varnothing\}$. Let t be the variable corresponding to the open set X, and let s be the variable corresponding to U. By construction, $S\colon X \to \Bbbk[s,t]$ is the function defined by $S(\eta) = s$, $S(\mathfrak{m}) = 0$, and likewise, $T\colon X \to \Bbbk[s,t]$ is defined by $T(\eta) = t$, $T(\mathfrak{m}) = t$. Then the ring A is given by $\Bbbk[S,T] \cong \Bbbk[s,t]$ (for now). We proceed to observe that $\Sigma(X) = \{(\eta,\eta), (\eta,\mathfrak{m}), (\mathfrak{m},\mathfrak{m})\}$.

Now, using Proposition 2.20, let us determine the pairs of elements $(a,b) \in A \times A$ for which $a \oslash b$ induces a v-extension of $\mathbf{A}$. Because $S(\eta), T(\eta) \neq 0$, we see that $\eta \in d(a)$ only if $a = 0$, in which case $a \oslash b = 0$ does not induce any v-extension. Thus, we need not consider $v_{\eta,\eta}$ and $v_{\eta,\mathfrak{m}}$; it suffices to consider the pairs (a,b) such that $\mathfrak{m} \in d(a)$, and $v_{\mathfrak{m},\mathfrak{m}}(A) \geq v_{\mathfrak{m},\mathfrak{m}}(b)$ with equality if and only if $\mathfrak{m} \in d(b)$. The condition $\mathfrak{m} \in d(a)$, i.e., $a(\mathfrak{m}) = 0$, is equivalent to the condition $a \in (S) \subset \Bbbk[S,T]$. The inequality $v_{\mathfrak{m},\mathfrak{m}}(A) \geq v_{\mathfrak{m},\mathfrak{m}}(b)$ holds only if $v_{\mathfrak{m},\mathfrak{m}}(b) = 0$, because the valuation $v_{\mathfrak{m},\mathfrak{m}}$ is by construction nonnegative on A, where $v_{\mathfrak{m},\mathfrak{m}}(t) = 0$. But $v_{\mathfrak{m},\mathfrak{m}}(b) = 0$ precisely when $b \in (S) \setminus \{0\}$, from the basic properties of a valuation. In short, we have found that $a \oslash b$ induces a v-extension only for $a, b \in (S)$, $b \neq 0$.

We obtain the indexed spring $C(\mathbf{A}, v)$ by adjoining the elements $\{a \oslash b \mid a, b \in (S), b \neq 0\}$. Moreover, under the isomorphism from $A = \Bbbk[S,T]$ to $A^\eta = \Bbbk[s,t]$ given by $a \mapsto a(\eta)$, we find that the ring of the extension $C(\mathbf{A}, v)$ is precisely $\Bbbk[s,t]$ localized at the ideal (s). We will abuse notation slightly and also denote this ring by A. Then $A = \Bbbk[s,t]_{(s)}$ is a discrete valuation ring, with unique nonzero maximal ideal (s), and we verify that $\operatorname{Spec} A = \{(0), (s)\}$ is homeomorphic to X as desired. □

Remark 2.38. While Hochster's construction is still feasible to compute for small spectral spaces, it quickly becomes unwieldly for any larger spaces, even relatively "simple" examples such as $\operatorname{Spec}\mathbb{Z}$ or $\operatorname{Spec}\Bbbk[t]$. We note that the construction *begins* with a polynomial ring with as many variables as there are sets in an open subbasis for X, then adjoins quotients on top of that. Thus, the construction is useful for its functorial properties and for demonstrating the *existence* of a ring whose spectrum is the given spectral space, but not necessarily for explicitly inverting Spec, which we do functorially in the following section.

3 Further consequences

3.1 Invertibility of Spec

Introducing indeterminates for a spectral space not only allows us to construct the functor taking a spectral space to its associated ring, but also provides a convenient method of inverting Spec, as we find below.

Theorem 3.1. Spec *is invertible on the following subcategories of* STop*.*

a. The subcategory of all spectral spaces and surjective spectral maps.

b. For each spectral space X*, the subcategory consisting of spectral subspaces of* X *and their inclusions.*

c. The full subcategory of all T_1 *spectral spaces. (Note that these are precisely the profinite spaces — the compact totally disconnected spaces.)*

Proof. Call the subcategory in question S.

a. Let $J\colon \mathsf{S} \to \mathsf{STopIt}$ be the functor sending each object $X \in \mathsf{S}$ to $(X, Q(X), \mathrm{id}_{Q(X)})$ and each morphism f in S to (f, f^{-1}). Note that since every f is surjective, f^{-1} is then injective, so J is indeed a space-preserving functor. The composition of J with $\mathrm{Spec} \circ F\colon \mathsf{STopIt} \to \mathsf{STop}$ is then naturally isomorphic to $\mathrm{id}_{\mathsf{STop}}$.

b. In this case, define J to be the functor sending each object Y to $(Y, Q(X), U \mapsto U \cap Y)$ and each morphism f to $(f, \mathrm{id}_{Q(X)})$.

c. This statement is actually a more elementary exercise in topology. We observe that $(X, \{\Bbbk\}_{x\in X}, \mathrm{C}(X, \Bbbk))$ is an affine spring, where X is a totally disconnected compact (i.e., quasicompact Hausdorff) space and $\mathrm{C}(X, \Bbbk)$ is the ring of continuous functions. It thus suffices to take the functor $X \mapsto \mathrm{C}(X, \Bbbk)$.

□

For the sake of completeness, we also find the following restrictions for inverting Spec on larger subcategories of STop, in particular focusing on certain types of inclusions of points and embeddings. This shows that the subcategories obtained in Theorem 3.1 are fairly maximal with respect to invertibility.

Proposition 3.2. Spec *is not invertible on the following examples of subcategories of* STop*.*

a. Subcategories containing a singleton space $\{*\}$*, a space* X*, and maps* $f\colon \{*\} \leftrightarrows X\colon g$ *such that* $f(*)$ *is not a closed point in* X*.*

b. Subcategories containing a singleton space $\{*\}$*, spaces* X, Y *with generic points, and distinct maps* $f\colon \{*\} \to Y$*,* $g, h\colon Y \rightrightarrows X$ *each preserving generic points.*

c. Subcategories containing a space X*, a family* $\{Y_i\}_{i\in I}$ *of spaces with generic points whose cardinalities are not bounded, and a family of maps (e.g., embeddings)* $\{f_i\colon X \to Y_i\}_{i\in I}$ *preserving generic points.*

d. For any space X *containing a point* p *whose closure contains at least three points, the subcategory of spectral subspaces of* X *and their embeddings.*

Proof. In each case, call the subcategory in question S, and suppose towards the sake of contradiction that Spec admits an inverse functor H from S to the category of reduced rings $\mathsf{Ring}_{\mathrm{red}}$.

a. By functoriality, we have that $H(f) \circ H(g) = \mathrm{id}_{H(*)}$, so $H(f)$ is a surjective ring homomorphism from $H(X)$ onto the field $H(*)$. Identifying $f\colon \{*\} \to X$ with $\mathrm{Spec}\, H(f)\colon \mathrm{Spec}\, H(*) \to \mathrm{Spec}\, H(X)$, it follows that $f(*)$ must be a closed point in X, a contradiction.

b. Observe that if f is a map preserving generic points, then $H(f)$ is an injective homomorphism (of reduced rings). Since the maps preserve generic points by assumption, we have that $H(f)$ is injective, and that $g \circ f = h \circ f$. By functoriality, $H(f) \circ H(g) = H(f) \circ H(h)$, which implies that $H(g) = H(h)$. But then $g = \mathrm{Spec}\, H(g) = \mathrm{Spec}\, H(h) = h$, a contradiction.

c. As in part (b), for each $i \in I$, $H(f_i)$ is an injective homomorphism from $H(Y_i)$ into $H(X)$. But the cardinalities of the rings $\{H(Y_i)\}_{i\in I}$ cannot be bounded, which is a contradiction: the ring $H(X)$ cannot have arbitrarily large cardinality.

d. Let q, r be two distinct points in $\mathrm{cl}\{p\} \setminus \{p\}$. Without loss of generality, we can suppose that $X = \{p, q, r\}$. Let Y be the spectral subobject $\{p, q\}$ of X. Then, in the given subcategory, there exists an inclusion map $f: \{p\} \to Y$, and two distinct maps $g, h: Y \rightrightarrows X$ preserving the generic point p such that $g(q) = q$ and $h(q) = r$. But then we have a contradiction by part (b): Spec cannot be invertible on this subcategory.

□

3.2 Equivalent characterizations of spectral spaces

In this section, we will explore how all spectral spaces can be obtained from the finite sober spaces.

Notation 3.3. Let $W = \{0, 1\}$ denote the Sierpiński space, in which 0 is a closed point and 1 is the generic point. (Cf. Example 2.37.) Equivalently, let W be up to homeomorphism the spectrum of a discrete valuation ring.

Notation 3.4. For a spectral space X, let $X^\vee$ denote $\mathrm{Hom}(X, W)$, the set of spectral maps from X to W.

Proposition 3.5. *A space X is spectral if and only if it is homeomorphic to a patch in a product of copies of W.*

Sketch of Proof. Note that a product of copies of W is spectral by the invertibility of Spec and the existence of tensor products in Ring. By straightforward verification of the topological properties, one also sees that patches of a spectral space are themselves spectral spaces. Conversely, let X be a spectral space. We can embed X as a subspace of $\prod_{f\in X^\vee} W$ through the standard evaluation map ev. Namely, define the image of $x \in X$ to be $\prod_{f\in X^\vee} \mathrm{ev}_x(f)$, where $\mathrm{ev}_x(f) = f(x)$. Equivalently, we can embed X into $\prod_{U\in Q(X)} W$ by sending each element $x \in X$ to $\prod_{U\in Q(X)} f_U(x)$, where $f_U(x) = 1$ if $x \in U$ and 0 otherwise. It follows that X is homeomorphic to its image. □

Theorem 3.6. *A space X is spectral if and only if it is the projective limit of finite sober spaces.*

Sketch of Proof. If X is spectral, by Proposition 3.2, we can realize X as a patch in the product $\Pi = \prod_{f\in X^\vee} W$. We can further express Π as a projective limit over all finite products in Π of copies of W. Then X is the projective limit of the images of X inside the finite subproducts of Π. The converse is much more straightforward and can be checked directly from the definition of a spectral space. □

3.3 Generalization to schemes

Finally, let us state and outline the proofs of generalizations of Theorem 1.3 to schemes. (Note that for historical reasons, "preschemes" and "schemes" in [3] are what we call "schemes" and "separated schemes" respectively.)

Definition 3.7. A **spectral subspace** is a subobject in STop, i.e., a spectral topological subspace whose inclusion map is a spectral map.

Theorem 3.8. *Let X be a topological space. The following are equivalent:*

1. *The space X is homeomorphic to the underlying space of a scheme S.*
2. *The space X is locally spectral: There exists an open cover of X by open spectral subspaces.*

The implication (1) $\implies$ (2) is a straightforward consequence of Theorem 1.3, so the bulk of the argument lies in the (functorial) construction of a scheme given a locally spectral space X. We will define a sheaf of rings on the basis $\mathscr{B}$ consisting of the open spectral subspaces of X as follows. Let B be the category whose objects are elements of $\mathscr{B}$, and whose morphisms are inclusion maps. We define a space-preserving functor $H: \mathsf{B} \to \mathsf{STopIt}$ by

$$H: Y \mapsto (Y, \{U \in \mathscr{B} \mid U \cap Y \in \mathscr{B}\}, U \mapsto U \cap Y).$$

If $F: \mathsf{STopIt} \to \mathsf{Ring}$ is the space-preserving functor from the proof of Theorem 1.3, then the composition $F \circ H$ is a space-preserving (contravariant) functor from B to Ring. In other words, $F \circ H$ is a presheaf on $\mathscr{B}$. We will let $\mathscr{O}_X$ be the sheafification of $F \circ H$. It remains to be checked that $(X, \mathcal{O}_X)$ is a scheme, but in fact one can find that for every open spectral subspace Y of X, the open subscheme $(Y, \mathcal{O}_{X \restriction Y})$ is affine [3, Theorem 9].

The picture is much clearer for separated schemes:

Theorem 3.9. *Let X be a topological space. The following are equivalent:*

1. *The space X is homeomorphic to the underlying space of a separated scheme S.*
2. *The space X is homeomorphic to an open subspace of a spectral space.*
3. *The space X is locally spectral and quasiseparated.*

Proof. The implication (1) $\implies$ (3) follows from Theorem 3.8 and the known fact that separated schemes are quasiseparated. The implication (2) $\implies$ (1) follows from the outlined proof of Theorem 3.8. Lastly, (3) $\implies$ (2) follows from considering the embedding of X into the spectral space $\prod_{f \in X^\vee} W$ from the proof of Theorem 3.6. Note that because X is locally spectral, the embedding is open as desired. □

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